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# Solutions of Non Linear Ordinary Differential Equations Determined by Phase Plane Methods

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*Abstract*
*Non linear differential equations occur while modeling the practical problems and determining the solution is not an easy task. The qualitative study of differential equations is concerned with how to deduce important characteristics of the solutions of differential equations without actually solving them. Many nonlinear effects in control systems, such as saturation, friction etc., are best approximated by linear segmented characteristics rather than continuous mathematical functions. Use of the phase plane to study the effects of this type of nonlinearity on the system response was one of the original contributions to the phase plane method in control. It has the advantage that it results in a phase plane divided up into different regions but with a linear differential equation describing the motion in each region.*

*Key words: Phase plane method, Non-linear ordinary Differential equation, simple pendulum,*

## Introduction

Ordinary differential equations can be used to model many different types of physical system. An ordinary differential equation (ODE) is an equation of the form

$$F(t, f, f', f'', \dots, f^{(N-1)}, f^{(N)})=0 \qquad \text{(I)}$$

(with $f^{(j)} = \frac{d^j f}{dt^j}$ $j = 1, \dots, N$ ) relating a function f(t) to its derivatives. The order of an ODE is the size of the highest derivative that appears, so equation (I) is $N^{th}$ order. An ODE is linear if it can be written as a linear combination of f and its derivatives, that is,

$$a_N(t)f^{(N)} + a_{N-1}(t)f^{(N-1)} + \dots + a_1(t)f' + a_0(t)f + b(t) = 0 \qquad \text{(II)}$$

Including possible addition of an inhomogeneous term b (t). All other ODEs, which are not of the form (II), are referred to as **nonlinear**.

More generally, one can consider systems of ODEs relating M functions $f_0, f_1, f_2, \dots, f_{M-1}$ and their derivatives of different orders. In fact, if a system of ODEs can be solved for the highest derivatives appearing (which generally requires the implicit function), then it can always be converted to a system of first-order equations .For example, suppose equation (I) can be solved explicitly for the $N^{th}$ derivative, as

$$f^{(N)} = F(t, f, f', f'', \dots, f^{(N-1)}) \qquad \text{(III)}$$

In that case, the single ODE (I) may be rewritten as the first-order system. In most applications of nonlinear ODEs, such as in the physical sciences or biology, they appear as coupled systems of either first or second order. For example, Newton's equations in mechanics (Arnold, 1989) are of the form

$$q'' = F(t, q, q')$$

relating the acceleration q″ of a particle (of unit mass) to the force F acting on it, which is a function of its position q and velocity q′, and possibly the time t. Mechanical systems are often derived from an action functional .In particular, if q′ is the highest derivative appearing in the Lagrangian, so that $L = L$(t, q, q′), then the corresponding Euler-Lagrange equations form a second-order system of ODEs.

For an $N^{th}$-order linear homogeneous ODE, of the form (II) with b≡0, the general solution is just a linear combination of N-independent solutions $s_j$, that is, with N arbitrary constants $A_1, \dots, A_N$. Ideally, one would like to express the general solution of an $N^{th}$-order nonlinear ODE as a function of $N$ arbitrary integration constants, but in general this is not possible. However, for systems (III) where the $F_j$ on the right-hand sides are suitably regular functions of all their arguments in the neighborhood of the initial data, the local existence of a solution to the initial value problem near $t=t_0$ can be proved by the Cauchy-Lipschitz method (Ince, 1926). In fact, whenever the $F_j$ are analytic functions, then the existence of a local solution to the initial value problem is guaranteed in some circle around $t=t_0$ in the complex $t$ plane (Hille, 1976). This means that, at least locally, the solution $f(t)$ can be considered as a function of the initial data $f(t_0), f'(t_0), f''(t_0), \dots, f^{(N-1)}(t_0)$ . The phase plane method was the first approach used by control engineers for studying the effects of nonlinearity in feedback control systems. The technique can generally only be used for systems represented by second order differential equations. It had previously been used in nonlinear mechanics and for the study of nonlinear oscillations. Smooth mathematical functions were assumed for the nonlinearities so that the second order equation could be represented by two nonlinear first order equations.

By introducing the phase plane methods I have considered nonlinear ordinary differential equation that will be determined by phase plane method. To analysis and demonstrate solutions of nonlinear ordinary differential equations that are determined by phase plane method. Obtain the transient behavior for some simple systems using the phase plane approach .To review some applications of nonlinear differential equation indicating the phase portrait.

## Methodology

In this report different related books available are examined and also as much as possible related topics are explored from the internet. From the collected materials as per my understanding I have tried to present it in easier approach. Then,

- A detail review of some phase plane method for non linear ordinary differential and many other related concepts has been made.
- Concepts of facts have been to find out different graphical interpretation of non -linear ordinary differential equation.

***Second Order Non Linear Ordinary Differential Equations***

Ordinary differential equations can be used to model many different types of physical system. We now know a lot about second order linear ordinary differential equations. For example, simple harmonic motion,

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \qquad (1)$$

describes many physical systems that oscillate with small amplitude $\theta$. The general solution is $\theta = A\sin \omega t + B \cos \omega t$, where $A$ and $B$ are constants that can be fixed from the initial values of $\theta$ and $d\theta/dt$ . The solution is an oscillatory function of $t$. Note that we can also write this as $\theta = Ce^{i\omega t}+De^{-i\omega t}$, where $C$ and $D$ are complex constants. In the real world, the physics of a problem is rarely as simple as this. Let's consider the frictionless simple pendulum, shown in Figure 4. A mass, $m$, is attached to a light, rigid rod of length $l$, which can rotate without friction about the point O.

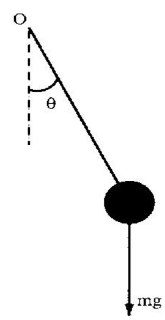


**Figure 1:** A simple pendulum.
(Source: A.C. King, J. Billingham and S.R. Otto, 2003. Differential Equations: Linear, Nonlinear, Ordinary, Partial. Cambridge University Press).

Using Newton's second law on the force perpendicular to the rod gives $-mg \sin \theta = ml \frac{d^2\theta}{dt^2}$ and hence

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0 \qquad (2)$$

where $\omega^2 = g/l$ and $t$ is time. For oscillations of small amplitude, $\theta \ll 1$, so that $\sin\theta \sim \theta$, and we obtain simple harmonic motion, (1). If $\theta$ is not small, we must study the full equation of motion, (2), which is **nonlinear**. In general, nonlinear ordinary differential equations cannot be solved analytically, but for equations like (2), where the first derivative, $d\theta/dt$ does not appear explicitly, an analytical solution is available. Using the notation $\dot\theta = d\theta/dt$, the trick is to treat $\dot\theta$ as a function of $\theta$ instead of $t$.

Note that

$$\frac{d^2\theta}{dt^2} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d\dot\theta}{dt} = \frac{d\theta}{dt}\frac{d\dot\theta}{d\theta} = \dot\theta\frac{d\dot\theta}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{2}\dot\theta^2\right)$$

This allows us to write (10) as $\frac{d}{d\theta}\left(\frac{1}{2}\dot\theta^2\right) = -\omega^2 \sin\theta$ which we can integrate once to give

$\frac{1}{2}\dot{\theta}^2 = \omega^2\cos\theta$+constant .Using $\omega^2 = \frac{g}{l}$ ,we can write this as

$$\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = E \qquad (3)$$

This is just a statement of conservation of energy, $E$, with the first term representing kinetic energy, and the second, gravitational potential energy. Systems like (2), which can be integrated once to determine a **conserved quantity**, here energy, $E$, are called **conservative systems**. Note that if we try to account for a small amount of friction at the point of suspension of the pendulum, we need to add a term proportional to $d\theta/dt$ to the left hand side of (2). The system is then no longer conservative, with dramatic consequences for the motion of the pendulum.

From (3) we can see that $\frac{d\theta}{dt} = \pm\sqrt{\frac{2E}{ml^2} + \frac{2g}{l}\cos\theta}$

We can integrate this to arrive at the implicit solution

$$t = \pm\int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{\frac{2E}{ml^2} + \frac{2g}{l}\cos\theta}} \qquad (4)$$

where $\theta_0$ is the angle of the pendulum when $t = 0$. Note that the two constants of integration are $E$ and $\theta_0$, the initial energy and angle of the pendulum. Equation (4) is a simple representation of the solution, which, if necessary, we can write in terms of Jacobian elliptic functions. Presumably equation (4) gives oscillatory solutions for small initial angles and kinetic energies, and solutions with $\theta$ increasing with $t$ for large enough initial energies. From (4) we have a quantitative expression for the solution, but we are really more interested in the qualitative nature of the solution. Let's go back to (3) and write

$$\dot{\theta}^2 = \frac{2E}{ml^2} + \frac{2g}{l}\cos\theta$$

Graphs of $\dot{\theta}^2$ as a function of $\theta$ are shown in Figure 5(a) for different values of $E$.

- For $E > mgl$ the curves lie completely above the $\theta$-axis.
- For $-mgl < E < mgl$ the curves intersect the $\theta$ - axis.
- For $E < -mgl$ the curves lie completely below the $\theta$ -a xis (remember, $-1 \leq \cos\theta \leq 1$).

We can now determine $\dot{\theta}$ as a function of $\theta$ by taking the square root of the curves in Figure 2(a) to obtain the curves in Figure 2(b), remembering to take both the positive and negative square root.

- For $E > mgl$, the curves lie either fully above or fully below the $\theta$ -axis.
- For $-mgl < E < mgl$, only finite portions of the graph o f $\dot{\theta}^2$ lie above the $\theta$ -axis, so the square root gives finite, closed curves.
- For $E < -mgl$, there is no real solution. This corresponds to the fact that the pendulum always has a gravitational potential energy of at least $-mgl$, so we must have $E \geq -mgl$.

As we shall see later, the solution with $E = mgl$ is an important one. The graph of $\dot{\theta}^2$ just touches the $\theta$-axis at $\theta = \pm(2n - 1)\pi$, for $n = 1, 2, \ldots,$ and taking the square root gives the curves that pass through these points shown in Figure 2(b). If $\dot{\theta}$ is positive, $\theta$ increases with $t$, and vice versa (remember, $\dot{\theta} = d\theta/dt$ is, by definition, the rate at which $\theta$ changes with $t$). This allows us to add arrows to Figure 2(b), indicating in which direction the solution changes with time. We have now constructed our first **phase portrait** for a nonlinear ordinary differential equation. The $(\theta, \dot{\theta})$ - plane is called the **phase plane**. Each of the solution curves represents a possible solution of (2), and is known as an **integral path** or **trajectory**. If we know the initial conditions,

$$\theta = \theta_0,\ \dot{\theta} = \dot{\theta}_0 = \sqrt{\frac{2E}{ml^2} + \frac{2g}{l}\cos\theta}$$

the integral path that passes through the point $(\theta_0, \dot{\theta}_0)$ when $t = 0$ represents the solution. Finally, note that, since $\theta = \pi$ is equivalent to $\theta = -\pi$, we only need to consider the phase portrait for $-\pi \leq \theta \leq \pi$. Alternatively, we can cut the phase plane along the lines $\theta = -\pi$ and $\theta = \pi$ and join them up, so that the integral paths lie on the surface of a cylinder.

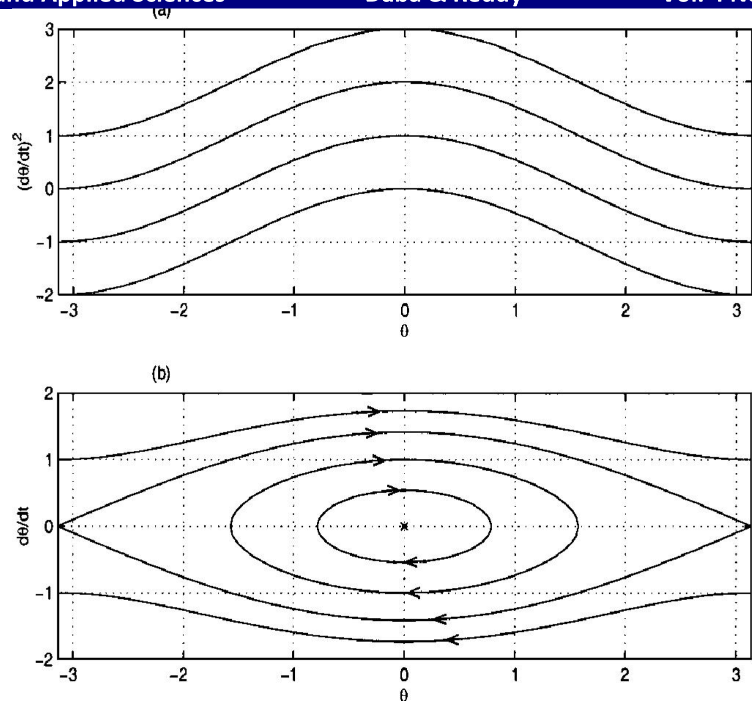


**Figure 2.** (a) $\dot{\theta}^2$ as a function of θ for different values of E. (b) $\dot{\theta}$ as a function of θ – the phase portrait for the simple pendulum. The cross at (0, 0) indicates an equilibrium solution. The arrows indicate the path followed by the solution as t **increases. Note that the phase portrait for |θ| ≥ *π* is the periodic extension of the phase portrait for |θ| ≤ *π*.**

*(Source: A.C. King, J. Billingham and S.R. Otto, 2003. Differential Equations: Linear, Nonlinear, Ordinary, Partial. Cambridge University Press).*

Let's consider the three qualitatively different types of integral path shown in Figure 5(b).

i. **Equilibrium solutions**: The points $\dot{\theta} = 0$, $\theta = 0$ or $\pm\pi$, represent the two equilibrium solutions of (10). The point (0, 0) is the equilibrium with the pendulum vertically downward, ($\pi$, 0) the equilibrium with the pendulum vertically upward. Points close to
(0, 0) lie on small closed trajectories close to (0, 0). This indicates that (0, 0) is a **stable equilibrium point**, since a small change in the state of the system away from equilibrium leads to solutions that remain close to equilibrium. If you cough on a pendulum hanging downwards, you will only excite a small oscillation. In contrast, points close to ($\pi$, 0) lie on trajectories that take the solution far away from ($\pi$, 0), and we say that this is an **unstable equilibrium point**. If you cough on a pendulum balanced precariously above its point of support, it will fall. Of course, in practice it is impossible to balance a pendulum in this way, precisely because the equilibrium is unstable.

ii. **Periodic solutions:** Integral paths with $-mgl < E < mgl$ are closed, and represent periodic solutions. They are often referred to as **limit cycles** or **periodic orbits**. The pendulum swings back and forth without reaching the upward vertical. These orbits are stable, since nearby orbits remain nearby. Note that the frequency of the oscillation depends upon its amplitude, a situation that is typical for nonlinear oscillators. For the simple pendulum, the amplitude of the motion is $\theta_{max} = \cos{-1}$ ($E/mgl$), and (4) shows that the period of the motion, $T$, is given by

$$T = \int_{-\theta_{max}}^{\theta_{max}} \frac{d\theta}{\sqrt{\frac{2E}{ml^2} + \frac{2g}{l}\cos\theta}}$$

Small closed trajectories in the neighborhood of the equilibrium point at (0, 0) are described by simple harmonic motion, (1). Note that, in contrast to the full nonlinear system, the frequency of simple harmonic motion is independent of its amplitude. The idea of **linearizing** a nonlinear system of ordinary differential equations close to an equilibrium point in order to determine what the phase portrait looks like there, is one that is important. In terms of the phase portrait on the cylindrical surface, trajectories with $E > mgl$ are also stable periodic solutions, looping round and round the cylinder. The pendulum has enough kinetic energy to swing round and round its point of support.

iii. **Heteroclinic solutions**: The two integral paths that connect ($-\pi$, 0) and ($\pi$, 0) have

$E = mgl$. The path with $\dot{\theta} \geq 0$ has $\theta \to \pm\pi$ as $t \to \pm\infty$ . This solution represents a motion where the pendulum swings around towards the upward vertical, and is just caught between falling back and swinging over. This is known as a **heteroclinic path**, since it connects different equilibrium points. Heteroclinic paths are important, because they represent the boundaries between qualitatively different types of behavior. They are also unstable, since nearby orbits behave qualitatively differently. Here, the heteroclinic orbits separate motions where the pendulum swings back and forth from motions where it swings round and round (different types of periodic solution). In terms of the phase portrait on a cylindrical surface, we can consider these paths to be **homoclinic paths**, since they connect an equilibrium point to itself.

We have now seen that, if we can determine the qualitative nature of the phase portrait of a second order nonlinear ordinary differential equation and sketch it, we can extract a lot of information about the qualitative behavior of its solutions. This information provides rather more insight than the analytical solution, (4), into the behavior of the physical system that the equation models. For non conservative equations, we cannot integrate the equation directly to get at the equation of the integral paths, and we have to be rather more cunning in order to sketch the phase portrait. We will develop methods to tackle second order, non conservative, ordinary differential equations.

***Some Applications of Non Linear Ordinary Differential Equations***

***An Example from Mechanics***

Consider a rigid block of mass $M$ attached to a horizontal spring. The block lies flat on a horizontal conveyor belt that moves at speed $U$ and tries to carry the block away with it, as shown in Figure 14. From Newton's second law of motion and Hooke's law,

$$M\ddot{x} = F(\dot{x}) - k(x - x_e),$$

where $x$ is the length of the spring, $x_e$ is the equilibrium length of the spring, $k$ is the spring constant, $F(\dot{x})$ is the frictional force exerted on the block by the conveyor belt, and a dot denotes $d/dt$. We model the frictional force as

$$F(\dot{x}) = \begin{cases} F_0 & for\ \dot{x} > U \\ -F_0 & for\ \dot{x} < U \end{cases}$$

with $F_0$ a constant force. When $\dot{x} = U$, the block moves at the same speed as the conveyor belt, and this occurs when $k|x - x_e| < F_0$. In other words, the force exerted by the spring must exceed the frictional force for the block to move. This immediately gives us a solution, $\dot{x} = U$ for

$x_e - F_0/k \leq x \leq x_e + F_0/k$.

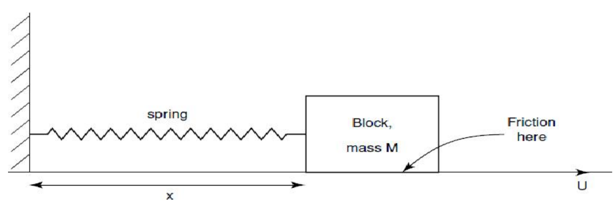


**Figure 3.** A spring-mounted, rigid block on a conveyor belt.

*(Source: King A.C., J. Billingham and S.R. Otto, 2003. Differential Equations: Linear, Nonlinear, Ordinary, Partial. Cambridge University Press)*

Our model involves five physical parameters, $M$, $k$, $x_e$, $F_0$ and $U$. If we now define dimensionless variables $\bar{x} = x/x_e$ and $\bar{t} = \frac{t}{\sqrt{M/k}}$ we obtain

$$\ddot{\bar{x}} = \bar{F} - \bar{x} + 1 \qquad (5)$$

where

$$\bar{F}(\dot{\bar{x}}) = \begin{cases} \bar{F}_0 & for\ \dot{\bar{x}} < \bar{U} \\ -\bar{F}_0 & for\ \dot{\bar{x}} > \bar{U} \end{cases} \qquad (6)$$

We also have the possible solution $\dot{\bar{x}} = \bar{U}$ for $1 - \bar{F}_0 \leq \bar{x} \leq 1 + \bar{F}_0$ . There are now just two dimensionless parameters,

$$\bar{F}_0 = \frac{F_0}{kx_e}, \quad \bar{U} = \frac{U}{x_e}\sqrt{M/k}$$

We can now write (23) as the system

$$\dot{x} = y, \quad \dot{y} = F(y) - x + 1 \qquad (7)$$

We have left out the over bars for notational convenience. This system has a single equilibrium point at $x = 1+F_0$, $y = 0$. Since $y = 0 < \bar{U}$ , the system is linear in the neighborhood of this point, with

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{F}_0 + 1 \end{pmatrix}$$

The Jacobian matrix has eigenvalues $\pm i$, so the equilibrium point is a linear centre. In fact, since $\dot{x}\,(x - 1 - F) + y\,\dot{y} = 0$, we can integrate to obtain

$$\{x - (1 + F_0)\}^2 + y^2 = \text{constant for } y < U,$$
$$\{x - (1 - F_0)\}^2 + y^2 = \text{constant for } y > U. \quad (8)$$

The solutions for $y \neq U$ are therefore concentric circles, and we conclude that the equilibrium point remains a centre when we take into account the effect of the nonlinear terms.

We now need to take some care with integral paths that meet the line $y = U$. Since the right hand side of (7) is discontinuous at $y = U$, the slope of the integral paths is discontinuous there.
For $x < 1 - F_0$ and $x > 1 + F_0$, trajectories simply cross the line $y = U$. However, we have already seen that the line $y = U$ for $1-F_0 \leq x \leq 1+F_0$ is itself a solution. We conclude that an integral path that meets $y = U$ with $x$ in this range follows this trajectory until $x = 1+F_0$, when it moves off on the limit cycle through the point $D$ in Figure 4. For example, consider the trajectory that starts at the point $A$. This corresponds to an initially stationary block, with the spring stretched so far that it can immediately overcome the frictional force. The solution follows the circular trajectory until it reaches $B$. At this point, the direction of the frictional force changes, and the solution follows a different circular trajectory, until it reaches $C$. At this point, the block is stationary relative to the conveyor belt, which carries it along with it until the spring is stretched far enough that the force it exerts exceeds the frictional force. This occurs at the point $D$. Thereafter, the solution remains on the periodic solution through $D$. On this periodic solution, the speed of the block is always less than that of the conveyor belt, so the frictional force remains constant, and the block undergoes simple harmonic motion.

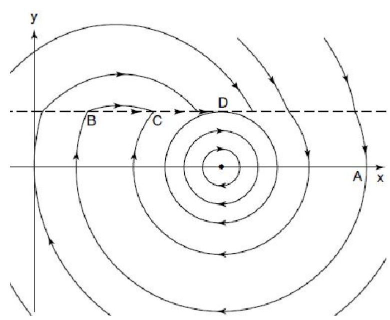


**Figure 4.** The phase portrait for a spring-mounted, rigid block on a conveyor belt.
(Source: King A.C., J. Billingham and S.R. Otto, 2003. Differential Equations: Linear, Nonlinear, Ordinary, Partial. Cambridge University Press)

## Conclusion

The qualitative study of differential equations is concerned with how to deduce important characteristics of the solutions of differential equations without actually solving them. Non linear ordinary differential equation occurs while modeling the practical problems and determining solution is not an easy task. We introduced a geometrical device, phase plane, which is used extensively for obtaining directly from differential equations such properties as equilibrium, periodicity, and stability and so on. The classical pendulum problem shows how the phase plane may be used to reveal the main features of the solutions of a particular differential. The solutions of the higher order system can be analyzed in terms of integral paths in a higher-dimensional, $n^{th}$-**phase space**. However, most of the useful results that hold for the phase plane do not hold in three or more dimensions. The crucial difference is that, in three or more dimensions, closed curves no longer divide the phase space into two distinct regions, inside and outside the curve. For example, integral paths inside a limit cycle in the phase plane are trapped there, but this is not the case in a three-dimensional phase space. The advantage of phase plane methods is to give a complete description of all solutions of systems of differential equations.

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