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Full Length Research Paper

Generalized Laplacian Operator for a Hydrogen Atom based upon the Riemannian Geometry

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Abstract
It is well known that many geometrical quantities in Cartesian and Spherical coordinates are built upon the Euclidean geometry for application to Physics and Mathematics. In this paper we derive the Laplacian operator for the Hydrogen atom based upon the Riemannian geometry.

Key words: *Riemannian geometry, Laplacian operator.*

Introduction

It is well known fact that since about 625 BC, all the laws of Physics and Mathematics are based upon the Euclidean geometry. In this geometry, the Laplacian operator in the Cartesian coordinates is given by,

$$\frac{d}{dx^2} + \frac{d}{dy^2} + \frac{d}{dz^2} = 0 \qquad (1)$$

And by the transformation,

$$X = r\sin\Theta\cos \qquad (2)$$
$$Y = r\sin\Theta\sin \qquad (3)$$
$$Z = r\cos\Theta \qquad (4)$$

Equation (1) may be written in spherical coordinates as,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] = 0 \qquad (5)$$

[Robert,1961]

The simple hydrogen atom

The hydrogen atom is a single electron atom whose well known Hamiltonian is give as,

$$H = \frac{P^2}{2m} - \frac{Ze^2}{r} \qquad (6)$$

For which the Hamiltonian operator is well known to be:

$$\hat{H} = \frac{-\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{r} \qquad (7)$$

Neglecting the potential part, equation (7) becomes,

$$\hat{H} = \frac{-\hbar^2}{2m}\nabla^2 \qquad (8)$$

Where; ∇^2 is the Euclidean Laplacian operator in spherical coordinates given as in equation (5)

Accordingly, the Schrödinger wave equation (SWE) in spherical coordinates on the Euclidean Geometry is given by, [Robert, 1961]

$$E\,(r,\Theta,\ ) = \frac{-\hbar^2}{2m}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\right\} + V(r) \qquad (9)$$

Where is any function.

The general solution of equation (9) is simply found by the simple method of separation variables given by,

$$(r,\Theta, \ ) = R(r)\rho(\Theta)F(\Phi) \tag{10}$$

It may be noted that equation (10) consist of:

(i) Radial solution which is a set of a polynomial series of the form of the associated Legendre polynomials L_n, l which only exist when n,l are integers number and $n \geq l$. The energy of each solution can be found by solving the radial Schrödinger equation, so that,

$$E_n = \frac{-\alpha^2 mc^2}{2n^2} \tag{11}$$

(ii) The solution of the angular part which are spherical harmonics

(iii) The azimuthal solution which give rise to different quantum numbers.

It is interesting to note here that, the Euclidean Laplacian given by equation (5) does not take into consideration the variation of proper mass with coordinate time. This anomaly is being corrected in the Riemannian Laplacian.

RIEMANNIAN LAPLACIAN OPERATOR ∇_R^2

The Riemannian Laplacian operator is defined as[*Howusu,* 2009,Spiegel,1974]

$$\nabla_R^2 = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}\left(\sqrt{g}\right)g^{\mu\upsilon}\frac{\partial}{\partial x^\upsilon}) \tag{12}$$

Where; $g_{\mu\upsilon}$ is a covariant metric tensor, g is the magnitude of the diagonal matrix of order **16.**

$$\mathbf{g} = \mathbf{g}_{11}g_{22}g_{33}g_{00} \tag{13}$$

where;

$$g_{11} = (1 + \frac{2}{c^2}f)^{-1} \tag{14}$$

$$g_{22} = r^2 (1 + \frac{2}{c^2}f) \tag{15}$$

$$g_{33} = (1 + \frac{2}{c^2}f)r^2\sin^2\Theta \tag{16}$$

$$g_{00} = - (1 + \frac{2}{c^2}f) \tag{17}$$

from (13) - (17),

$$g = - r^4\sin^2\Theta (1 + \frac{2}{c^2}f)^2 \tag{18}$$

and

$$\sqrt{|g|} = r^2\sin\Theta (1 + \frac{2}{c^2}f) \tag{19}$$

It follows that the Riemannian Laplacian can be written more explicitly in spherical polar coordinates as,

$$\nabla_R^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left\{\left(1+\frac{\epsilon}{c^2}f\right)r^2\frac{\partial}{\partial r}\right\} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{c^2}\frac{\partial}{\partial t}\left\{\left(1+\frac{\epsilon}{c^2}f\right)^{-1}\frac{\partial}{\partial t}\right\} \tag{20}$$

Where f is the gravitational scalar potential on the electron due to the earth or planet and is given as,

$$f = \frac{-k}{R} \quad ; r = R \tag{21}$$

and

$$k = GM_E \tag{22}$$

and

$$\epsilon = 2$$

Substituting equation (21) and (22) and the value of ϵ in equation (20) we have,

$$\nabla_R^2 = \frac{1}{r^2}\left\{\left(1-\frac{2k}{c^2 R}\right)\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right\} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{c^2}\left\{\left(1-\frac{2k}{c^2 R}\right)^{-1}\frac{\partial^2}{\partial t^2}\right\} \tag{23}$$

Equation (23) is the explicit form of the generalized Riemannian Laplacian operator for a hydrogen atom in spherical coordinates [HOwusu, 2013]

Conclusion

The Riemannian Laplacian operator equation (23) reduces to the pure Euclidean in the limit of c^0 and in general contains post Euclidean or pure Riemannian correction terms of all orders of c^{-2} and all orders of non linearity in the gravitational scalar potential f.

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