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Full Length Research Paper

Quantum Theory of the Linear Harmonic Oscillator Based upon General Quantum Mechanical Wave Equation

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Abstract
It is well-known that the eigenenergies of the Linear Quantum Simple Harmonic Oscillator can be obtain by using Schrodinger wave equation that the foundation is based upon Euclidean Geometry. In this paper, we apply the General Quantum Mechanical Wave Equation that the foundation is based upon Riemannian geometry to the linear harmonic oscillator

Keywords: *General mechanical wave equation, Eigenenergies and gravitational potential energy.*

Introduction

The Schrodinger's solution of the Quantum Linear simple Harmonic Oscillator is built upon the Euclidean metric tensor. This metric tensor is the basis foundation for the derivation of Laplacian operator called the Euclidean Laplacian used in the quantum mechanical wave equation for the linear simple harmonic oscillator problems before now. Follows the introduction of a new metric tensor called the great metric tensor by Professor S.X.KHowusu, in the book entitled: Riemannian Revolution in Physics and Mathematics [1], we formulate a new Laplacian operator called the Riemannian Laplacian in Cartesian coordinate based upon the great metric tensors. This Riemannian Laplacian is used in the formulation of the General Quantum Mechanical Wave Equation for application in the Quantum theory of Linear Simple Harmonic Oscillator.

Methodology

According to the book entitled '*Riemannian Revolutions in Physics and Mathematics'* the General Quantum Mechanical Wave Equation for a particle of nonzero rest mass m_0 is given in all orthogonal curvilinear coordinates x^u by [1]:

$$i\hbar\frac{\partial}{\partial t}\psi(x^u) = \left\{-\frac{\hbar^2}{2m_0}\nabla_R^2 + V_R + V_{ng}\right\}\psi(x^u)(1)$$

Where V_R is Riemann's gravitational energy given by [1]:

$$V_R(x^u) = -\frac{1}{2}m_0 g_{00}(\dot{x})^2 + \frac{1}{2}m_0 c^2 (2)$$

And ∇_R^2 is the Riemannian Laplacian operator given by [2]

$$\nabla_R^2(x^v) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^v}\left\{\sqrt{g}g^{\mu v}\frac{\partial}{\partial x^v}\right\} \quad (3)$$

And V_{ng} is the non-gravitational potential energy of the particle, $g_{\mu v}$ is the metric tensor, and $\hbar$ is Dirac's constant.

Next in Einstein Cartesian Co-ordinates (x^0, x, y, z) the one and only one (unique) metric tensor for all gravitational fields in nature is given by [3]:

$$g_{00} = \left(1 + \frac{2}{c^2}f\right) (4)$$

$$g_{11} = (1 + f_{11})(5)$$

$$g_{22} = (1 + f_{22})(6)$$

$$g_{00} = (1 + f_{33})(7)$$

$g_{12} = g_{21} = f_{12} = f_{21}$ (8)

$g_{13} = g_{31} = f_{13} = f_{31}$ (9)

$g_{\mu\nu} = 0; otherwise$ (10)

Where;

$$f_{11} = \frac{x^2}{(x^2+y^2+z^2)} \sum_{n=1}^{\infty} \binom{-1}{n} \frac{2^n}{c^{2n}} f^n \quad (11)$$

$$f_{22} = \frac{y^2}{(x^2+y^2+z^2)} \sum_{n=1}^{\infty} \binom{-1}{n} \frac{2^n}{c^{2n}} f^n \quad (12)$$

$$f_{11} = \frac{z^2}{(x^2+y^2+z^2)} \sum_{n=1}^{\infty} \binom{-1}{n} \frac{2^n}{c^{2n}} f^n \quad (13)$$

$$f_{12} = \frac{xy}{(x^2+y^2+z^2)} \sum_{n=1}^{\infty} \binom{-1}{n} \frac{2^n}{c^{2n}} f^n \quad (14)$$

$$f_{13} = \frac{xz}{(x^2+y^2+z^2)} \sum_{n=1}^{\infty} \binom{-1}{n} \frac{2^n}{c^{2n}} f^n \quad (15)$$

$$f_{23} = \frac{yz}{(x^2+y^2+z^2)} \sum_{n=1}^{\infty} \binom{-1}{n} \frac{2^n}{c^{2n}} f^n \quad (16)$$

Next in the case of the terrestrial harmonic oscillator the Riemann's gravitational potential energy is given by [1]:

$$V_R = m_0 f + m_0 \sum_{n=1}^{\infty} \binom{-1}{n} \frac{2^n}{c^{2n}} f^n \quad (17)$$

Where; f is the gravitational scalar potential given by

$$f = -\frac{GM_0}{R} \quad (18)$$

Where; M_0 *and* R are the mass and radius of the earth respectively and G is the universal gravitational constant, and

$$V_R = \frac{1}{2} m_0 \omega_0^2 (x^2+y^2+z^2) \quad (19)$$

And ω_0 is the natural frequency.

Next specializing all the results above to the one-dimensional harmonic oscillator along the x-axis and approximating all results correct to the order of c^{-2},we obtain the General Quantum Mechanical Wave Equation given by:

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m_0}\left\{(1+\epsilon_1)^{-1}\frac{\partial^2}{\partial x^2} + \frac{1}{c^2}(1+\epsilon_2)^{-1}\frac{\partial^2}{\partial t^2}\right\}\psi(x,t) + \left\{V_R + \frac{1}{2}m_0\omega_0^2 x^2\right\}\psi(x,t) \quad (20)$$

Where;

$$\epsilon_1 = \sum_{n=1}^{\infty} \binom{-1}{n} \frac{2^n}{c^{2n}} f^n \quad (21)$$

And

$$\epsilon_1 = \frac{2}{c^2} f \quad (22)$$

And f is given by (18). This equation reveals some of the post Schrodinger correction terms of the General quantum Mechanical Wave Equation.

To separate the variable in (20) let us seek the solution of the form

$$\psi(x,t) = X(x) \exp\left(\frac{-iEt}{\hbar}\right) \quad (23)$$

Where E is the energy of the oscillator and X is the energy wave function. Then we obtain the energy wave equation as:

$$0 = -\frac{\hbar^2}{2m_0}(1+\epsilon_1)^{-1}X''(x) + \left\{\frac{1}{2m_0c^2}(1+\epsilon_2)^{-2}E^2 - E + V_R + \frac{1}{2}m_0\omega_0^2 x^2\right\}X(x) \quad (24)$$

In the first case let us neglect ϵ_1 and ϵ_2 in (24). Then we obtain the energy wave equation as

$$0 = -\frac{\hbar^2}{2m_0}X''(x) + \left\{\left[\frac{1}{2m_0c^2}E^2 - E + V_R\right] + \frac{1}{2}m_0\omega_0^2 x^2\right\} \quad (25)$$

Results

It is perfectly clear that this equation may be solved exactly as in the case of the corresponding pure Schrodinger's energy wave equation to obtain the energy eigenvalues given by the equation [4, 5]:

$$\frac{1}{2m_0c^2}E^2 - E + V_R = -\left(n+\frac{1}{2}\right)\hbar\omega_0; n = 1,2,\ldots \quad (26)$$

Or equivalently

$$0 = \frac{E_n^2}{2m_0} - E_n + \left[V_R + \left(n+\frac{1}{2}\right)\hbar\omega_0\right]; n = 1,2,\ldots \quad (27)$$

Or equivalently

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega_0 + V_R + \sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}\frac{(-1)^k 2^k}{m_0^{k-1}c^{2(k-1)}}\left[\left(n+\frac{1}{2}\right)\hbar\omega_0 + V_R\right]; n = 1,2,\ldots \quad (28)$$

Therefore the General Quantum Mechanical Wave Equation has extended the pure Schrodinger's quantum mechanical eigenenergies to include the gravitational potential energy of the oscillator. It has also generated hitherto unknown, but mathematically most sound and elegant and physically most natural and radical and satisfactory corrections of all order of c^{-2} to the pure Schrodinger's quantum mechanical eigenenergies and hence eigenfunctions of the linear simple harmonic oscillator. In this also the corresponding energy wave functions are given by

$$X_n(\xi) = N_n \exp\left(-\frac{1}{2}\xi^2\right)H_n(\xi) \quad (29)$$

Where; H_n is the Hermite polynomial of order n and N_n is the normalization constant given by

$$N_n = \left(2^n n!\sqrt{\pi}\right)^{\frac{1}{2}} (30)$$

And

$$\xi = \left(\frac{m_0\omega_0}{\hbar}\right)^{\frac{1}{2}} x \quad (31)$$

Conclusion

In the paper we showed how to formulate the General Quantum Mechanical Wave Equation in the Cartesian Coordinates and apply it to the linear harmonic oscillator. It is now obvious how to formulate the General Quantum Mechanical Wave Equation in any of the orthogonal curvilinear co-ordinates and apply it to any physical problem.

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