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Some Upper Bounds of Super Edge-Magic Deficiency of Chain Graphs

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Abstract
R. M. Figuroa-Centeno et al. brought out the idea of super edge-magic deficiency, $\mu_s(G)$, of simple graphs which is the minimum integer $n \in N$ {0} for which $G \cup nK_1$ admits a super edge-magic labeling or it is $+\infty$ if no such n exists. In this article, we examine upper bounds of super edge-magic deficiency of two classes of chain graphs.

***Keywords**: Super edge-magic deficiency, chain graphs.*

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Introduction
We are dealing here with finite, simple and undirected graphs. We denote vertex and edge sets of a graph G by $V(G)$ and $E(G)$ respectively. The (p, q)-graph G is a graph having $|V(G)| = p$ and $|E(G)| = q$. A (p, q)-graph G is said to admit an edge-magic labeling if it admits a bijection $\delta: V(G) \cup E(G) \rightarrow \{1, 2, . . . , p+q\}$ such that $\delta(x) + \delta(xy) + \delta(y) = c$ is a constant (called magic constant, magic sum or valence of G under δ), for each edge $xy \in E(G)$. An edge-magic labeling of G becomes super edge-magic if it has the additional characteristic that $\delta(V(G)) = \{1, 2, . . . , p\}$. A graph that admits a super edge-magic labeling is said to be super edge-magic. Graph labeling has caught attention of many mathematicians which is not only due to mathematical problems that are being called into question in order to obtain accuracy in this area, but also for its wide application in astronomy, coding, circuit and network designing. Long ago, G. S. Bloom and S. W. Golomb studied applications of graph labeling to various branches of science in their articles, some of their discussions can be seen in [1, 2].

Primarily, H. Enomoto *et al.* [3] gave the idea of super edge-magic labeling of graphs. Figueroa-Centeno *et al.* [6] showed that if G is a super edge-magic bipartite or tripartite graph, then for odd m, mG is super edge-magic. In [3] H. Enomoto *et al*. proved a complete bipartite graph $K_{m,n}$ to be super edge-magic if and only if $m = 1$ or $n = 1$. In [6] it is proved that $K_{1,m} \cup K_{1,n}$ is super edge-magic if either m is a multiple of $n + 1$ or n is a multiple $m + 1$. H. Enomoto *et al.* [3] proved that C_n is super edge-magic if and only if n is odd. $C_3 \cup C_n$ is super edge-magic [9] if and only if $n \geq 6$ and n is even (also see [10]). Graph theorists are still working on this famous conjecture. In [3] H. Enomoto *et al.* showed that the generalized prism $C_{2m+1} \times P_m$ is super edge-magic for every positive integer m (also see [4]). The following G lemma is quite interesting as far as super edge-magic graphs are concerned.

Lemma 1. [4] A (p, q)-graph G is super edge-magic if and only if there exists a bijective function $\delta : V(G) \rightarrow \{1, 2, \cdots, p\}$ such that the set $S = \{\delta(x) + \delta(y) | \ xy \in E(G)\}$ consists of q consecutive integers. In such a case, δ extends to a super edge-magic labeling of G with magic constant $c = p + q + s$, where $s = min(S)$ and $S = \{c - (p + 1), c - (p + 2), . . . , c - (p + q)\}$.

Kotzig and A. Rosa [11] proved that for every graph G there exists an edge-magic graph H such that $H \cong G \bigcup nK_1$ for some non-negative integer n. This fact leads to the concept of edge-magic deficiency $\mu(G)$ of a graph G, which is the minimum non-negative integer n for which $G \cup nK_1$ is an edge-magic graph. Figueroa-Centeno *et al.* [5] defined a similar concept for super edge-magic graphs. The super edge-magic deficiency of a graph G, denoted by $\mu_s(G)$, is the minimum non-negative integer n such that $G \cup nK_1$ admits a super edge-magic labeling or $+\infty$ if no such integer n *exists*. *i.e.*, if

$$M(G) = \{n \geq 0 : G \cup nK_1 \text{ is a super edge-magic graph}\}$$

Then;

$$\mu_s(G) = \begin{cases} \min M(G) : M(G) \neq \Phi \\ \\ +\infty : M(G) = \Phi \end{cases}$$

In [5, 8] Figueroa-Centeno *et al.* have given exact values of the super edge-magic deficiencies of several families of graphs, such as cycles, complete graphs, 2-regular graphs and complete bipartite graphs $K_{2,m}$. They also proved that all forests have finite deficiency. In [13] A. Ngurah, Simanjuntak and Baskoro gave upper bounds for the super edge-magic deficiency of fans, double fans and wheels. In [7] Figueroa-Centeno *et al*. conjectured that every forest with two components has super edge-magic deficiency at most 1.

Concluding the introductory talk, super edge-magic deficiency of many standard graphs have been discussed by the graph theorists in the past. Discussion on the super edge-magic deficiency on their disjoint union is also highly considerable, which is base of this article as well.

Results

In this article, we deal with super edge-magic deficiency of some planar graphs which form chain type structure. In fact, we have calculated upper bounds of super edge-magic deficiency of these graphs.

Definition 1. Let H_n be the chain graph consisting of n copies of C_5 having vertex set:

$$V(H_n) = \{x_1, x_{2:} : 1 \le i \le n+1\} \cup \{z_i : 1 \le i \le n\}$$

and edge set

$$E(H_n) = \{z_i y_{i,} \, z_i y_{i+1} : 1 \le i \le n\} \cup \{x_i y_i 1 \le i \le n+1\} \cup \{x_i x_{i+1:} \, 1 \le i \le n+1\}.$$

Where $|V(H_n)| = 3n + 2$ and $|E(H_n)| = 4n + 1$.

In the upcoming result, we show an upper bound for super edge-magic deficiency of graph H_n.

Theorem 1. For the graph H_n;

$$\mu(H_n) = \begin{cases} = 0 : n \equiv 1,2; \\ \\ \le \frac{n}{2} : n \equiv 0 (\mathrm{mod}\, 2); \\ \\ \le \frac{n-1}{2} : n \equiv 1 (\mathrm{mod}\, 2) \end{cases}$$

Proof. For $n = 1$, $H_1 \cong C_5$ is clearly super edge-magic [5]. For $n = 2$, consider the labeling set of H_2 as $\{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2$: 6, 7, 5, 1, 2, 3, 4, 8$\}$

- For $2 < n \equiv 0 \pmod 2$:

Consider the graph $H' \cong H_n \cup (\frac{n}{2})K_1$ with $|V(H')| = \frac{1}{2}(7n + 4)$. Also, $V(H') = V(H_n) \cup \{d_i : 1 \le i \le \frac{n}{2}\}$. We define a labeling $f : V(\underline{H'}) \to \{1, 2, 3, \ldots, \frac{1}{2}(7n + 4)\}$ as:

$$f(x_i) = \begin{cases} \frac{1}{2}(i+1) : 1 \le i \le n+1; i \equiv 1 (\mathrm{mod}\, 2); \\ \frac{1}{2}(3n+i+10) : 2 \le i \le n; i \equiv 0 (\mathrm{mod}\, 2). \end{cases}$$

$$f(z_i)=\begin{cases}\frac{1}{2}(5n+10):i=1;\\ \frac{1}{2}(7n-2i+8):2\le i\le \frac{n}{2}-1;\\ \frac{1}{2}(n+4):i=\frac{n}{2};\\ \frac{1}{2}(3n+5):i=\frac{n}{2}+1;\\ \frac{1}{2}(7n-2i+12):\frac{n}{2}+2\le i\le n.\end{cases}$$

$$f(d_i)=\begin{cases}\frac{1}{2}(3n+8):i=1;\\ 2n+i+4:2\le i\le \frac{n}{2}.\end{cases}$$

$$f(y_i)=\frac{1}{2}(3n-2i+8):1\le i\le n+1.$$

Set of all edge sums generated by above labeling scheme form a consecutive integer sequence $n+4, n+5, n+6, \dots, 5n+4$. Therefore by Lemma 1, H' is super edge-magic with magic constant $\frac{1}{2}(17n+14)$;

$$\Rightarrow \mu_s(H_n)\le \frac{n}{2}.$$

- For $1 < n \equiv 1 (\text{mod } 2)$:

Consider the graph $H'' \cong H_n \cup (\frac{n-1}{2})K_1$ with $|V(H'')| = \frac{1}{2}(7n+3)$. Also, $V(H'') = V(H_n) \cup \{d_i : 1 \le i \le \frac{n-1}{2}\}$. We define a labeling $g: V(H'') \rightarrow \{1, 2, 3, \dots, \frac{1}{2}(7n+3)\}$ as:

$$g(x_i)=\begin{cases}\frac{1}{2}(i+1):1\le i\le n; i\equiv 1(\text{mod } 2);\\ \frac{1}{2}(3n+i+3):2\le i\le n+1; i\equiv 0(\text{mod } 2).\end{cases}$$

$$g(y_i)=\frac{1}{2}(3n-2i+5):1\le i\le n+1;$$

$$g(z_i)=\frac{1}{2}(7n-2i+5):1\le i\le n;$$

$$g(d_i)=2n+i+2:1\le i\le \frac{n-1}{2}.$$

Set of all edge sums generated by above labeling scheme g form a consecutive integer sequence $n + 3, n + 4, n + 5, \ldots, 5n + 3$. Under the light of Lemma 1, H'' is super edge-magic with magic constant $\frac{1}{2}(17n + 11)$;

$$\Rightarrow \mu_s(H_n) \leq \frac{n-1}{2}.$$

Here we are in a position to propose the following open problem for further working.

Open Problem 1. Determine a sharper upper bound or the exact value of super edge-magic deficiency of H_n.

Definition 2. Consider a graph E_n of order $3(n+1)$ and size $5n+2$ consisting of vertex set;

$$V(E_n) = \{x_i, y_i : 1 \leq i \leq n+1\} \cup \{z_i : 1 \leq i \leq n\} \cup \{c\}$$

and edge set;

$$E(E_n) = \{z_i y_i, z_i y_{i+1} : 1 \leq i \leq n\} \cup \{x_i y_i : 1 \leq i \leq n+1\} \cup \{x_i x_{i+1} : 1 \leq i \leq n\}$$
$$\cup \{c y_i : 1 \leq i \leq n+1\}.$$

In next theorem, we address an upper bound of super edge-magic deficiency of E_n.

Theorem 2. For the graph E_n;

$$\mu(E_n) = \begin{cases} n : n \equiv 0 (\mathrm{mod}\, 2); \\ n-1 :: n \equiv 1 (\mathrm{mod}\, 2) \end{cases}$$

Proof. We discuss two cases simultaneously,

- For $n \equiv 0 \pmod 2$:

Consider the graph $E' \cong E_n \cup nK_1$ with $|V(E')| = 4n + 3$. Also, $V(E') = V(E_n) \cup \{d_i : 1 \leq i \leq n\}$. Define a labeling $f : V(E') \to \{1, 2, 3, \ldots, 4n + 3\}$ as:

$f(c) = 1$;

$$f(x_i) = \begin{cases} \frac{1}{2}(n+i+3) : 1 \leq i \leq n+1; i \equiv 1 (\mathrm{mod}\, 2); \\ \frac{1}{2}(4n+i+12) : 2 \leq i \leq n; i \equiv 0 (\mathrm{mod}\, 2). \end{cases}$$

$$f(z_i) = \begin{cases} 3n+6 : i = 1; \\ 4n-i+5 : 2 \leq i \leq \frac{n}{2} - 1; \\ n+3 : i = \frac{n}{2}; \\ 2(n+3) : i = \frac{n}{2} + 1; \\ 4n-i+7 : \frac{n}{2} + 2 \leq i \leq n. \end{cases}$$

$$f(d_i) = \begin{cases} i+1 : 1 \leq i \leq \frac{n}{2}; \\ 2n+5 : i = \frac{n}{2} + 1; \\ 2n+i+5 : \frac{n}{2} + 2 \leq i \leq n. \end{cases}$$

$f(y_i) = 2n - i + 5 : 1 \le i \le n+1$.

Set of all edge sums generated by above labeling scheme form a consecutive integer sequence $n + 5, n + 6, n + 7, \ldots, 6(n + 1)$. Therefore by Lemma 1, E' is super edge-magic with magic constant $10(n + 1)$.

$$\Rightarrow \mu_s(H_n) \le n. \quad \text{for even } n.$$

- For $n \equiv 1 (\text{mod } 2)$:

Consider the graph $E'' \cong E_n \cup n(n-1)K_1$ with $|V(E'')| = 2(2n + 1)$. Also, $V(E'') = V(E_n) \cup \{d_i: 1 \le i \le n-1\}$. We define a labeling $g : V(E'') \to \{1, 2, 3, \ldots, 2(2n + 1)\}$ as:

$$g(x_i) = \begin{cases} \frac{1}{2}(n+i+1) : 1 \le i \le n; i \equiv 1(\text{mod } 2); \\ \frac{1}{2}(4n+i+4) : 2 \le i \le n+1; i \equiv 0(\text{mod } 2). \end{cases}$$

$$g(y_i) = 2n - i + 3 : 1 \le i \le n+1;$$

$$g(z_i) = 4n - i + 3 : 1 \le i \le n;$$

$g(c) = 1$

$$f(d_i) = \begin{cases} i+1 : 1 \le i \le \frac{n-1}{2}; \\ 2n+i+3 : \frac{n+1}{2} \le i \le n-1. \end{cases}$$

Set of all edge sums generated by above labeling scheme g form a consecutive integer sequence $n + 3, n + 4, n + 5, \ldots, 6n + 4$. Lemma 1 implies E'' is super edge-magic with magic constant $10n + 7$;

$$\Rightarrow \mu_s(E_n) \le n - 1, \quad \text{for odd } n.$$

Open Problem 2. Determine a sharper upper bound or the exact value of super edge-magic deficiency of E_n.

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