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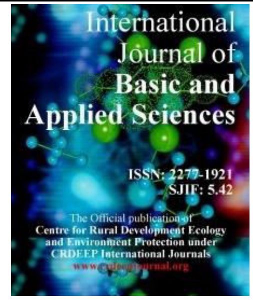

Full Length Research Paper

On the Deficiency of Disjoint Union of Multicyclic and Acyclic Graphs

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ABSTRACT

R. M. Figuroa-Centeno et al. brought out the idea of super edge-magic deficiency, $\mu_s(G)$, of simple graphs which is the minimum integer $n \in N \{0\}$ for which $G \cup nK_1$ admits a super edge-magic labeling or it is $+\infty$ if no such n exists. In this article, we study the super edge-magic deficiency of disjoint unions of graphs mainly multicyclic structures with trees and forests.

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Introduction

Throughout this article, we deal with finite, simple and undirected graphs. We denote vertex and edge sets of a graph G by $V(G)$ and $E(G)$ respectively. The (p, q)-graph G is a graph having $|V(G)| = p$ and $|E(G)| = q$. A (p, q)-graph G is said to admit an edge-magic labeling if it admits a bijection $\delta: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ such that $\delta(x) + \delta(xy) + \delta(y) = c$ is a constant (called magic constant, magic sum or valence of G under δ), for each edge $xy \in E(G)$. An edge-magic labeling of G becomes super edge-magic if it has the additional characteristic that $\delta(V(G)) = \{1, 2, \ldots, p\}$. A graph that admits a super edge-magic labeling is said to be super edge-magic. Graph labeling has caught attention of many mathematicians which is not only due to mathematical problems that are being called into question in order to obtain accuracy in this area, but also for its wide application in astronomy, coding, circuit and network designing. Long ago, G. S. Bloom and S. W. Golomb studied applications of graph labeling to various branches of science in their articles, some of their discussions can be seen in [1, 2].

Primarily, H. Enomoto *et al.* [3] gave the idea of super edge-magic labeling of graphs. Figueroa-Centeno *et al.* [6] showed that if G is a super edge-magic bipartite or tripartite graph, then for odd m, mG is super edge-magic. In [3] H. Enomoto *et al.* proved a complete bipartite graph $K_{m,n}$ to be super edge-magic if and only if $m = 1$ or $n = 1$. In [6] it is proved that $K_{1,m} \cup K_{1,n}$ is super edge-magic if either m is a multiple of $n + 1$ or n is a multiple $m + 1$. H. Enomoto *et al.* [3] proved that C_n is super edge-magic if and only if n is odd. $C_3 \cup C_n$ is super edge-magic [9] if and only if $n \geq 6$ and n is even (also see [10]). Graph theorists are still working on this famous conjecture. In [3] H. Enomoto *et al.* showed that the generalized prism $C_{2m+1} \times P_m$ is super edge-magic for every positive integer m (also see [4]). The following G lemma is quite interesting as far as super edge-magic graphs are concerned.

Lemma 1. [4] A (p, q)-graph G is super edge-magic if and only if there exists a bijective function $\delta : V(G) \rightarrow \{1, 2, \cdots, p\}$ such that the set $S = \{\delta(x) + \delta(y) | \, xy \in E(G)\}$ consists of q consecutive integers. In such a case, δ extends to a super edge-magic labeling of G with magic constant $c = p + q + s$, where $s = min(S)$ and $S = \{c - (p + 1), c - (p + 2), \ldots, c - (p + q)\}$.

Kotzig and A. Rosa [11] proved that for every graph G there exists an edge-magic graph H such that $H \cong G \bigcup nK_1$ for some non-negative integer n. This fact leads to the concept of edge-magic deficiency $\mu(G)$ of a graph G, which is the minimum non-negative integer n for which $G \cup nK_1$ is an edge-magic graph. Figueroa-Centeno *et al.* [5] defined a similar concept for super edge-magic graphs. The super edge-magic deficiency of a graph G, denoted by $\mu_s(G)$, is the minimum non-negative integer n such that $G \cup nK_1$ admits a super edge-magic labeling or $+\infty$ if no such integer n exists. *i.e.*, if

$M(G) = \{n \geq 0 : G \cup nK_1$ is a super edge-magic graph$\}$

then

$$\mu_s(G) = \begin{cases} \min M(G) : M(G) \neq \Phi \\ +\infty : M(G) = \Phi \end{cases}$$

In [5, 8] Figueroa-Centeno *et al.* have given exact values of the super edge-magic deficiencies of several families of graphs, such as cycles, complete graphs, 2-regular graphs and complete bipartite graphs $K_{2,m}$. They also proved that all forests have finite deficiency. In [13] A. Ngurah, Simanjuntak and Baskoro gave upper bounds for the super edge-magic deficiency of fans, double fans and wheels. In [7] Figueroa-Centeno *et al.* conjectured that every forest with two components has super edge-magic deficiency at most 1.

Concluding the introductory talk, super edge-magic deficiency of many standard graphs have been discussed by the graph theorists in the past. Discussion on the super edge-magic deficiency on their disjoint union is also highly considerable, which is base of this article as well.

Results

In the present article, we will calculate some upper bounds and exact values of the super edge-magic deficiency of a multicyclic family of graphs with various trees and forest.

Definition 1. Consider a multicyclic graph G_m, where m is parameter used in the cycle C_m, with connection scheme as follows;

$$V(G_m) = \{x_1^i, x_2^i : 1 \leq i \leq m\} \cup \{y_i, z_i : 1 \leq i \leq m\}$$

$$E(G_m) = \{y_i y_{i+1}, z_i z_{i+1} : 1 \leq i \leq m-1\} \cup \{y_1 y_m, z_1 z_m\}$$
$$\cup \{y_i x_1^i, y_i x_2^i : 1 \leq i \leq m\} \cup \{z_i x_1^i, z_i x_2^i : 1 \leq i \leq m\}.$$

In our first result, we shall calculate an upper bound of the super edge-magic deficiency of the above multicyclic graphs with star $K_{1,m}$.

Theorem 1. For odd m, the super edge-magic deficiency of disjoint union of G_m with $K_{1,m}$ is at most $\frac{m-1}{2}$.

Proof. Consider a graph $G_1 \cong G_m \cup K_{1,m} \cup (\frac{m-1}{2})K_1$ with vertex and edge sets:

$$V(G_1) = \{x_1^i, x_2^i : 1 \leq i \leq m\} \cup \{y_i, z_i : 1 \leq i \leq m\}$$
$$\cup \{k_i : 1 \leq i \leq m\}\{d_i : 1 \leq i \leq \frac{m-1}{2}\} \cup \{c\}.$$

$$E(G_1) = \{y_i y_{i+1}, z_i z_{i+1} : 1 \leq i \leq m-1\} \cup \{y_1 y_m, z_1 z_m\}$$
$$\cup \{y_i x_1^i, y_i x_2^i : 1 \leq i \leq m\} \cup \{z_i x_1^i, z_i x_2^i : 1 \leq i \leq m\}$$
$$\cup \{ck_i : 1 \leq i \leq m\}.$$

Then $p = |V(G_1)| = \frac{11m+1}{2}$ and $q = |E(G_1)| = 7m$. Now, define a labeling $f : V(G_1) \to \{1, 2, ..., \frac{11m+1}{2}\}$ as follows:

$$f(x_1^i) = \begin{cases} \frac{1}{2}(5m+i) : 1 \leq i \leq m; i \equiv 1 (\text{mod } 2); \\ \frac{1}{2}(4m+i) : 2 \leq i \leq m-1; i \equiv 0 (\text{mod } 2). \end{cases}$$

$$f(y_i) = \begin{cases} \frac{1}{2}(i+2m+1) : 1 \leq i \leq m; i \equiv 1 (\text{mod } 2); \\ \frac{1}{2}(i+3m+1) : 2 \leq i \leq m-1; i \equiv 0 (\text{mod } 2). \end{cases}$$

$$f(z_i) = \begin{cases} \frac{1}{2}(i+8m+1) : 1 \leq i \leq m; i \equiv 1 (\text{mod } 2); \\ \frac{1}{2}(i+9m+1) : 2 \leq i \leq m-1; i \equiv 0 (\text{mod } 2). \end{cases}$$

$$f(x_2^i) = 4m - i + 1 : 1 \leq i \leq m.$$

$$f(k_i) = i : 1 \leq i \leq m.$$

$$f(c) = p;$$

$$f(d_i) = 5m + i : 1 \leq i \leq \frac{m-1}{2}.$$

The set of all edge-sums generated by the above scheme forms a consecutive integer sequence $\frac{5m+3}{2}, \frac{5m+5}{2}, ..., \frac{19m+1}{2}$. Therefore by Lemma 1, f extends to a super edge-magic labeling of G_1 with magic constant $15m+2$.

We propose the following open problem for further work here.

Open Problem 1. *Determine a sharper upper bound or the exact value of* $\mu_s(G_m \cup K_{1,m})$.

Open Problem 2. *For even m, can you find an upper bound or the exact value of super edge-magic deficiency of* $G_m \cup K_{1,m}$*?*

In our next result, we shall calculate the exact value of super edge-magic deficiency of disjoint union of G_m with a forest consisting of m copies of P_2.

Theorem 2. *For odd m, the exact value of super edge-magic deficiency of disjoint union of G_m with mP_2 is zero.*

Proof. Consider a graph $G_2 \cong G_m \cup mP_2$ for odd m with following vertex and edge connection:

$$V(G_2) = \{x_1^i, x_2^i : 1 \le i \le m\} \cup \{y_i, z_i : 1 \le i \le m\}$$
$$\cup \{p_i, q_i : 1 \le i \le m\}.$$

$$E(G_2) = \{y_i y_{i+1}, z_i z_{i+1} : 1 \le i \le m-1\} \cup \{y_1 y_m, z_1 z_m\}$$
$$\cup \{y_i x_1^i, y_i x_2^i : 1 \le i \le m\} \cup \{z_i x_1^i, z_i x_2^i : 1 \le i \le m\}$$
$$\cup \{p_i q_i : 1 \le i \le m\}.$$

Then $p = |V(G_2)| = 6m$ and $q = |E(G_2)| = 7m$. Now, define a labeling $g : V(G_2) \to \{1,2,...,6m\}$ as follows:

$$g(x_1^i) = \begin{cases} \frac{1}{2}(5m+i) : 1 \le i \le m; i \equiv 1(\bmod 2); \\ \frac{1}{2}(4m+i) : 2 \le i \le m-1; i \equiv 0(\bmod 2). \end{cases}$$

$$g(y_i) = \begin{cases} \frac{1}{2}(i+2m+1) : 1 \le i \le m; i \equiv 1(\bmod 2); \\ \frac{1}{2}(i+3m+1) : 2 \le i \le m-1; i \equiv 0(\bmod 2). \end{cases}$$

$$g(z_i) = \begin{cases} \frac{1}{2}(i+8m+1) : 1 \le i \le m; i \equiv 1(\bmod 2); \\ \frac{1}{2}(i+9m+1) : 2 \le i \le m-1; i \equiv 0(\bmod 2). \end{cases}$$

$$g(z_i) = \begin{cases} \frac{1}{2}(i+8m+1) : 1 \le i \le m; i \equiv 1(\bmod 2); \\ \frac{1}{2}(i+9m+1) : 2 \le i \le m-1; i \equiv 0(\bmod 2). \end{cases}$$

$$g(x_2^i) = 4m - i + 1 : 1 \le i \le m.$$

$$g(p_i) = i : 1 \le i \le m.$$

The set of all edge-sums generated by the above scheme forms a consecutive integer sequence $\frac{5m+3}{2}, \frac{5m+5}{2}, ..., \frac{19m+1}{2}$. Therefore by Lemma 1, f extends to a super edge-magic labeling of G_2 with magic constant $\frac{31m+3}{2}$.

Open Problem 2. *For even m, can you find an upper bound or the exact value of super edge-magic deficiency of $G_m \cup K_{1,m}$?*

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