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Full Length Research Paper

A Discussion on Strongly *k*-Indexable Lattices

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Abstract

B. D. Acharya and S. M. Hedge defined the concept of strongly k-indexable graphs and relevant theory in their articles. On the other hand, the lattices are gaining repute due to its applications in molecular graph theory. The work on both of these ideas is presentably fruitful for researchers. Combining both ideas, the article in hands also covers k- indexability of lattices for possible values of k.

Keywords: *k- indexible, strongly k- indexible, lattice.*

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Introduction

The finite, simple and undirected graphs are only consideration in this study. We are denoting vertex and edge sets of a graph G by $V(G)$ and $E(G)$ respectively. The definitions and theory we are mentioning first are directly important in the presentation of our work. Primarily, a (p, q)-graph G is a graph having $|V(G)| = p$ and $|E(G)| = q$. A (p, q)-graph G is said to admit an edge-magic labeling if it admits a bijection $\delta\colon V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ such that $\delta(x) + \delta(xy) + \delta(y) = c$ is a constant, for each edge $xy \in E(G)$. An edge-magic labeling of G becomes super edge-magic if it has the additional characteristic that $\delta(V(G)) = \{1, 2, \ldots, p\}$. A graph that admits a super edge-magic labeling is said to be super edge-magic. The credit of these concepts goes to Hikoe Enomoto *et al* [5]. G. S. Bloom and S. W. Golomb studied applications of graph labeling to various branches of science in their articles, some of their discussions can be seen in [3, 4].

Figueroa-Centeno *et al.* [8] showed that if G is a super edge-magic bipartite or tripartite graph, then for odd m, mG is super edge-magic. In [5] H. Enomoto *et al*. proved a complete bipartite graph $K_{m,n}$ to be super edge-magic if and only if $m = 1$ or $n = 1$. In [8] it is proved that $K_{1,m} \cup K_{1,n}$ is super edge-magic if either m is a multiple of $n + 1$ or n is a multiple $m + 1$. H. Enomoto *et al.* [5] proved that C_n is super edge-magic if and only if n is odd. $C_3 \cup C_n$ is super edge-magic [11] if and only if $n \geq 6$ and n is even (also see [10]). Graph theorists are still working on this famous conjecture. In [5] H. Enomoto *et al.* showed that the generalized prism $C_{2m+1} \times P_m$ is super edge-magic for every positive integer m (also see [6]).

Lemma 1. [6] A (p, q)-graph G is super edge-magic if and only if there exists a bijective function $\delta : V(G) \rightarrow \{1, 2, \cdots, p\}$ such that the set $S = \{\delta(x) + \delta(y) | \; xy \in E(G)\}$ consists of q consecutive integers. In such a case, δ extends to a super edge-magic labeling of G with magic constant $c = p + q + s$, where $s = min\,(S)$ and $S = \{c - (p + 1), c - (p + 2), \ldots, c - (p + q)\}$.

Moving forward, if one studies [1] and [2], the following concepts can be seen. An additive numbering of a graph $G = (V, E)$ is an injective additive vertex function f such that the induced edge function $f+$ is also injective. A graph G for which $\theta(G) = |V(G) - 1|$ is said to be indexable and any minimal numbering of such a graph will be called an indexer. An additive numbering f of a (p, q)-graph G will be called a strong indexer if $f(G) = \{0, 1, 2, \ldots, p - 1\}$ and $f+(G) = \{1, 2, \ldots, q\}$. If G admits such a numbering it is called strongly indexable.

A (p, q)-graph G is said to be strongly k- indexable if its vertices can be assigned distinct integers $0, 1, 2, \ldots, p - 1$ so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices can be arranged as an arithmetic progression $k, k + 1, k + 2, \ldots, k + (q - 1)$. A (p, q)-graph G is said to be (k, d)-arithmetic if its vertices can be assigned distinct nonnegative integers so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices, can be arranged in the arithmetic progression $k, k + d, k + 2d, \ldots, k + (q - 1)d$. Concluding the introductory discussion, we are mainly going to present here some well designed infinite lattices and term them as M_1, M_2, M_3 and M_4 and focus on presenting their k-indexability.

Main Results
The following theorems present our main working, definition and results.

Theorem 1. The infinite lattice M_1 is strongly *3*- indexable for all possible value of *m.*

Proof. We define first the infinite lattice M_1 with following vertex and edge sets,

$$V(M_1)=\{x_i, y_i : 1\le i\le 2m\}\cup\{c_i : 1\le i\le m\}.$$

$$\begin{aligned}E(M_1)=&\{c_iy_{i+1}, : 1\le i\le 2m-1, i\equiv 1(\text{mod } 2)\}\cup\{c_iy_i, : 1\le i\le 2m-1, i\equiv 1(\text{mod } 2)\}\\ &\cup\{c_iy_{i-1}, : 2\le i\le 2m, i\equiv 0(\text{mod } 2)\}\cup\{c_ix_{i+1}, : 1\le i\le 2m-1, i\equiv 1(\text{mod } 2)\}\\ &\cup\{x_ix_{i+1}, y_iy_{i+1} : 2\le i\le 2(m-1), i\equiv 0(\text{mod } 2)\}\cup\{c_iy_i, : 2\le i\le 2m, i\equiv 0(\text{mod } 2)\}\\ &\cup\{c_ix_i, : 2\le i\le 2m-1, i\equiv 1(\text{mod } 2)\}\cup\{c_ix_{i-1}, : 2\le i\le 2m, i\equiv 0(\text{mod } 2)\}\\ &\cup\{c_ix_i, : 2\le i\le 2m, i\equiv 0(\text{mod } 2)\}\cup\{c_ic_{i+1}, : 2\le i\le 2m-1\}.\end{aligned}$$

Then $p=|V(M_1)|=6m$ and $q=|E(M_1)|=3(4m-1)$. We are defining a bijective function on M_1 as $f: V(M_1)\rightarrow\{1,2,\ldots,6m\}$ as follows:

$$f(x_i)=\begin{cases}3i : 1\le i\le 2m-1; i\equiv 1(\text{mod } 2);\\ 3i-1 : 2\le i\le 2m; i\equiv 0(\text{mod } 2).\end{cases}$$

$$f(y_i)=\begin{cases}3i-1 : 1\le i\le 2m-1; i\equiv 1(\text{mod } 2);\\ 3i-2 : 2\le i\le 2m; i\equiv 0(\text{mod } 2).\end{cases}$$

$$f(c_i)=\begin{cases}3i-2 : 1\le i\le 2m-1; i\equiv 1(\text{mod } 2);\\ 3i : 2\le i\le 2m; i\equiv 0(\text{mod } 2).\end{cases}$$

The edge-sums generated by the above scheme form a sequence of consecutive integers $3,4,\ldots,12m-1$. Thus with appropriate edge- labels (assigned in an opposite order) f refers that M_1 is strongly k-indexable for k=3.

Theorem 2. The infinite lattice M_2 is strongly *3*- indexable for all possible value of *m.*

Proof. We define first the infinite lattice M_2 with following vertex and edge sets,

$$V(M_2)=\{x_i, y_i : 1\le i\le 2m\}\cup\{c_i : 1\le i\le m\}$$
$$\cup\{u_i, v_i : 1\le i\le m\}.$$

$$E(M_2)=\{c_iy_{i+1},:1\le i\le 2m-1,i\equiv 1(\text{mod }2)\}\cup\{c_iy_i,:1\le i\le 2m-1,i\equiv 1(\text{mod }2)\}$$
$$\cup\{c_iy_{i-1},:2\le i\le 2m,i\equiv 0(\text{mod }2)\}\cup\{c_ix_{i+1},:1\le i\le 2m-1,i\equiv 1(\text{mod }2)\}$$
$$\cup\{x_ix_{i+1},y_iy_{i+1}:2\le i\le 2(m-1),i\equiv 0(\text{mod }2)\}\cup\{c_iy_i,:2\le i\le 2m,i\equiv 0(\text{mod }2)\}$$
$$\cup\{c_ix_i,:2\le i\le 2m-1,i\equiv 1(\text{mod }2)\}\cup\{c_iu_{\frac{i+1}{2}},c_iv_{\frac{i+1}{2}}:2\le i\le 2m-1,i\equiv 0(\text{mod }2)\}$$
$$\cup\{c_ix_{i-1},:2\le i\le 2m,i\equiv 0(\text{mod }2)\}\cup\{c_iu_{\frac{i}{2}},c_iv_{\frac{i}{2}}:2\le i\le 2m,i\equiv 0(\text{mod }2)\}$$
$$\cup\{c_ix_i,:2\le i\le 2m,i\equiv 0(\text{mod }2)\}\cup\{c_ic_{i+1},:2\le i\le 2m-1\}.$$

Then $p=|V(M_2)|=8m$ and $q=|E(M_2)|=16m-3$. We are defining a bijection on M_2 as $g:V(M_2)\to\{1,2,...,8m\}$ as follows:

$$g(x_i)=4i-1:1\le i\le 2m-1;i\equiv 1(\text{mod }2);$$

$$g(y_i)=\begin{cases}4i-2:1\le i\le 2m-1;i\equiv 1(\text{mod }2);\\ 4i-6:2\le i\le 2m;i\equiv 0(\text{mod }2).\end{cases}$$

$$g(c_i)=\begin{cases}4i-3:1\le i\le 2m-1;i\equiv 1(\text{mod }2);\\ 4i:2\le i\le 2m;i\equiv 0(\text{mod }2).\end{cases}$$

$$g(u_i)=8i-3:1\le i\le m;$$

$$g(v_i)=8i-4:1\le i\le m;$$

The edge-sums generated by the above scheme form a sequence of consecutive integers $3,4,...,12m-1$. Thus with appropriate edge- labels (assigned in an opposite order) g refers that M_2 is strongly k-indexable for k=3.

Theorem 3. The infinite lattice M_3 is strongly *3*- indexable for all possible value of *m.*

Proof. We define first the infinite lattice M_3 with following vertex and edge sets,

$$V(M_3)=\{x_i,y_i:1\le i\le 2m\}\cup\{c_i:1\le i\le 2(2m-1)\}$$
$$\cup\{u_i,v_i:1\le i\le m-1\}.$$

$$E(M_3)=\{c_iy_{\frac{i+2}{2}},:2\le i\le 2(2m-1),i\equiv 2(\text{mod }4)\}\cup\{c_iy_{\frac{i}{2}}:2\le i\le 2(2m-1),i\equiv 2(\text{mod }4)\}$$
$$\cup\{c_iy_{\frac{i+1}{2}},:1\le i\le 4m-3,i\equiv 1(\text{mod }4)\}\cup\{c_iy_{\frac{i+3}{2}},:1\le i\le 4m-3,i\equiv 2(\text{mod }4)\}$$
$$\cup\{c_ix_{\frac{i+2}{2}},:2\le i\le 2(2m-1),i\equiv 2(\text{mod }4)\}\cup\{c_ix_{\frac{i}{2}}:2\le i\le 2(2m-1),i\equiv 2(\text{mod }4)\}$$
$$\cup\{c_ix_{\frac{i+1}{2}},:1\le i\le 4m-3,i\equiv 1(\text{mod }4)\}\cup\{c_ix_{\frac{i+3}{2}},:1\le i\le 4m-3,i\equiv 2(\text{mod }4)\}$$
$$\cup\{c_ic_{i+1},:2\le i\le 4(m-1),i\equiv 0(\text{mod }4),i\equiv 2(\text{mod }4)\}\cup\{c_ic_{i+1},:1\le i\le 4m-3,i\equiv 1(\text{mod }4)\}$$
$$\cup\{u_ic_{4i},v_ic_{4i}:1\le i\le m-1\}\cup\{u_ic_{4i-1},v_ic_{4i-1}:1\le i\le m-1\}\cup\{u_iv_i:1\le i\le m-1\}$$

Then $p=|V(M_3)|=10m-4$ and $q=|E(M_3)|=20m-11$. We are defining a bijection on M_3 as $h:V(M_3)\rightarrow\{1,2,\ldots,10m-4\}$ as follows:

$$h(x_i)=\begin{cases}5i-2:1\leq i\leq 2m-1; i\equiv 1(\text{mod } 2);\\ 5(i-1):2\leq i\leq 2m; i\equiv 0(\text{mod } 2).\end{cases}$$

$$h(y_i)=\begin{cases}5i-3:1\leq i\leq 2m-1; i\equiv 1(\text{mod } 2);\\ 5i-6:2\leq i\leq 2m; i\equiv 0(\text{mod } 2).\end{cases}$$

$$h(c_i)=\begin{cases}\frac{1}{2}(5i-3):1\leq i\leq 4m-3; i\equiv 1(\text{mod } 4);\\ \frac{1}{2}(5i+2):1\leq i\leq 4m-3; i\equiv 1(\text{mod } 4);\\ \frac{1}{2}(5i-1):3\leq i\leq 4m-5; i\equiv 3(\text{mod } 4);\\ \frac{1}{2}(5i):4\leq i\leq 4(m-1); i\equiv 0(\text{mod } 4).\end{cases}$$

$$h(u_i)=10i-1:1\leq i\leq m-1;$$

$$h(v_i)=10i-2:1\leq i\leq m-1;$$

The edge-sums generated by the above scheme form a sequence of consecutive integers $3,4,\ldots,20m-9$. Thus with appropriate edge- labels (assigned in reverse order) h refers that M_3 is strongly *3*-indexable.

Theorem 4. The infinite lattice M_4 is strongly *3*- indexable for all possible value of *m.*

Proof. We define the infinite lattice M_4 with following vertex and edge sets,

$$V(M_4)=\{x_i,y_i:1\leq i\leq 2m\}\cup\{c_i:1\leq i\leq 2(2m-1)\}$$
$$\cup\{u_i,v_i:1\leq i\leq m\}\cup\{m_i,n_i:1\leq i\leq m\}.$$

$$E(M_4)=\{c_iy_{\frac{i+2}{2}},: 2\le i\le 2(2m-1), i\equiv 2(\text{mod }4)\}\cup\{c_iy_{\frac{i}{2}} : 2\le i\le 2(2m-1), i\equiv 2(\text{mod }4)\}$$
$$\cup\{c_iy_{\frac{i+1}{2}},: 1\le i\le 4m-3, i\equiv 1(\text{mod }4)\}\cup\{c_iy_{\frac{i+3}{2}},: 1\le i\le 4m-3, i\equiv 2(\text{mod }4)\}$$
$$\cup\{c_ix_{\frac{i+2}{2}},: 2\le i\le 2(2m-1), i\equiv 2(\text{mod }4)\}\cup\{c_ix_{\frac{i}{2}}, x_iu_{\frac{i}{2}}, y_iv_{\frac{i}{2}} : 2\le i\le 2(2m-1), i\equiv 2(\text{mod }4)\}$$
$$\cup\{c_ix_{\frac{i+1}{2}},: 1\le i\le 4m-3, i\equiv 1(\text{mod }4)\}\cup\{c_ix_{\frac{i+3}{2}},: 1\le i\le 4m-3, i\equiv 2(\text{mod }4)\}$$
$$\cup\{c_ic_{i+1},: 2\le i\le 4(m-1), i\equiv 0(\text{mod }4), i\equiv 2(\text{mod }4)\}\cup\{c_ic_{i+1},: 1\le i\le 4m-3, i\equiv 1(\text{mod }4)\}$$
$$\cup\{u_ic_{4i}, v_ic_{4i} : 1\le i\le m-1\}\cup\{u_ic_{4i+1}, v_ic_{4i+1} : 1\le i\le m-1\}\cup\{u_iv_i : 1\le i\le m-1\}$$
$$\cup\{u_ix_{2i+1}, v_iy_{2i+1} : 1\le i\le m-1\}$$

Then $p=|V(M_4)|=10m-4$ and $q=|E(M_4)|=20m-11$. We are defining a bijection on M_3 as $l:V(M_3)\to\{1,2,\ldots,10m-4\}$ as follows:

$$l(x_i)=\begin{cases}6i-3:1\le i\le 2m-1; i\equiv 1(\text{mod }2);\\ 6i-5:2\le i\le 2m; i\equiv 0(\text{mod }2).\end{cases}$$

$$l(y_i)=\begin{cases}6i-4:1\le i\le 2m-1; i\equiv 1(\text{mod }2);\\ 6(i-1):2\le i\le 2m; i\equiv 0(\text{mod }2).\end{cases}$$

$$l(c_i)=\begin{cases}3i-2:1\le i\le 4m-3; i\equiv 1(\text{mod }4);\\ 3i+2:1\le i\le 4m-2; i\equiv 2(\text{mod }4);\\ 6i:3\le i\le 4m-5; i\equiv 3(\text{mod }4);\\ 6i:4\le i\le 4(m-1); i\equiv 0(\text{mod }4).\end{cases}$$

$$l(u_i)=12i-1:1\le i\le m-1;$$

$$l(v_i)=12i-2:1\le i\le m-1;$$

$$l(n_i)=12i-8:1\le i\le m;$$

The edge-sums generated by the above scheme form a sequence of consecutive integers $3,4,\ldots,12(2m-1)$. Thus with appropriate edge- labels (assigned in reverse order) l refers that M_4 is strongly *3*-indexable.

Conclusion

A (p, q)-graph G is said to be strongly k- indexable if its vertices can be assigned distinct integers 0, 1, 2, . . . ,$p-1$ so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices can be arranged as an arithmetic progression $k, k+1, k+2, \ldots, k+(q-1)$. In the present article, we have discussed strong k - indexability of lattices M_1, M_2, M_3, M_4 for k= 3, and for rest of the possible values, it is open for others.

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