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On (*k, d*)- Arithmetic Convex Polytopes

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Abstract
A (p, q) graph G is said to be (k, d)-arithmetic if its vertices can be assigned distinct nonnegative integers so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices, can be arranged in the arithmetic progression k, k + d, k + 2d.. . . , k + (q - 1)d. In this article, we are providing some (k, d)- arithmetic convex polytopes for possible values of k and d.

Keywords: *(k, d)- arithmetic, convex polytope.*

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Introduction

We are considering finite, simple and undirected graphs. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively. A (p, q)-graph G is one with $|V(G)| = p$ and $|E(G)| = q$. Moreover, the theoretic ideas of graphs can be seen in [21]. A labeling (or valuation) of a graph is in fact a mapping that carries elements of graph to numbers (usually to positive or non-negative integers). Here, the domain will be $V(G) \cup E(G)$. In other words, the labeling in this article is total labeling. In some labelings only the vertex set or the edge set will be used and we shall call them vertex-labeling or edge-labelings, respectively. Graph labelings has many types such as harmonius, radio, cordial, graceful and antimagic. The recent survey of graph labelings can be seen in [5]. In this paper, we will focus on antimagic total labeling type. In [1], more details on an antimagic total labeling can be seen. The notion of edge-magic total labeling of graphs derives its origin in the research work of A. Kotzig and A. Rosa [12, 13] for which they used the terminology magic valuation. Let us now move to few useful definitions and relevant research work previously done.

Definition 1. A (k, d) edge-antimagic vertex ($(k, d)-$EAV) labeling of a graph G is a bijection $\rho: V(G) \to \{1, 2, \cdots, p\}$ such that the set of edge-sums of all edges in G, $\{w(xy) = \rho(x) + \rho(y) : xy \in E(G)\}$, forms an arithmetic progression $\{k, k + d, k + 2d, \ldots, k + (q - 1)d\}$, where $k > 0$ and $d \geq 0$ are fixed integers.

R. Simanjuntak *et al*., [19] proved that cycles and path, C_{2n+1}, P_{2n+1} and P_{2n}, have an $(n + 2, 1)$-EAV labeling when $n \geq 1$. They further proved that the odd path P_{2n+1} has a $(n + 3, 1)$-EAV labeling and the path P_n admits a $(3, 2)$-EAV labeling for $n \geq 1$. In [3], M. Baca *et al*., proved that if a connected graph G (must not be a tree) has an (a, d)-EAV labeling then $d = 1$. Further that a cycle C_n has no (a, d)-EAV labeling for $d > 1$ and $n \geq 3$ [3].

Definition 2. A (k, d) edge-antimagic total labeling of a graph G is a bijection $\rho: V(G) \cup E(G) \to \{1, 2, \cdots, p + q\}$ such that the set of edge-weights of all edges in G, $\{w(xy) = \rho(x) + \rho(xy) + \rho(y) : xy \in E(G)\}$, forms an arithmetic progression $\{k, k+d, k+ 2d, \ldots, k+ (q - 1)d\}$, where $k > 0$ and $d \geq 0$ are fixed integers. The graph G, if admits such labeling, is called an (k, d) edge-antimagic total graph. (abbreviated as $(a, d) -$ EAT labeling/ graph)

Definition 3. An (k, d)-EAT labeling ρ is called a super (k, d) edge antimagic total labeling of G if $\rho(V(G)) \to \{1, 2, \cdots, v\}$. Thus, a super (k, d)- edge-antimagic total graph is a graph that admits a super (k, d) edge-antimagic total labeling. (abbreviated as super $(k, d) -$ EAT labeling/ graph). If $d = 0$, then a super $(k, 0)$-EAT labeling is called a super edge-magic total labeling and k is called a magic constant or valence. For d other than 0, k is called minimum edge-weight. These all (k, d) definitions related graphs are also termed as (k, d) arithmetic graphs.
The definition of an (k, d)- EAT labeling was established by R. Simanjuntak, Bertault and M. Miller in [19] as a natural extension of an edge-magic total labeling defined by A. Kotzig and A. Rosa earlier. A super (k, d)- EAT labeling is a further natural extension of the notion of a super $(k, 0)$-EAT labeling introduced by Hikoe Enomoto *et al.*, in [4]. And not to forget the following interesting conjecture of the same paper that every tree admits a super $(k, 0)$ edge-antimagic total labeling. Many researchers have pillared this conjecture by deriving super (k, d)- EAT labeling for many particular classes of trees. As

in, stars, path like trees, *W*-trees, subdivided stars, caterpillars and lobsters. All such results can be seen in [2, 6, 7, 9, 10, 11, 8, 15, 17, 18, 20]. And the famous computer calculated verification of upto 17 vertices tree [14].

Main Results

Our central discussion in this paper is (*k, d*)- arithmetic labeling of some convex polytopes and some multicyclic graphs for possible values of *k* and appropriate values of *d.* We shall define the polytope in its corresponding theorem.

Theorem 1. The convex polytope CP_1 *is* $(\frac{m-1}{2}, d)$ and $(\frac{m+3}{2}, d)$ - arithmetic .

Proof. We define first the infinite polytope CP_1 with following vertex and edge sets,

$$V(CP_1) = \{u_i^j : 1 \le i \le m, 1 \le j \le l, j \equiv 1 (\text{mod } 2)\}$$
$$\cup \{v_{1i}^j, v_{2i}^j, v_{3i}^j, v_{4i}^j, v_{5i}^j : 1 \le i \le m, 1 \le j \le l-1, j \equiv 0 (\text{mod } 2)\}.$$

$$E(CP_1) = \{v_{1i}^j u_i^{j+1}, v_{2i}^j u_i^{j+1}, v_{3i}^j u_i^{j+1}, v_{4i}^j u_i^{j+1}, v_{5i}^j u_i^{j+1} : 1 \le i \le m, 1 \le j \le l-1, j \equiv 0 (\text{mod } 2)\}$$
$$\cup \{v_{1i}^{j+1} u_i^j, v_{2i}^{j+1} u_i^j, v_{3i}^{j+1} u_i^j, v_{4i}^{j+1} u_i^j, v_{5i}^{j+1} u_i^j : 1 \le i \le m, 1 \le j \le l-2, j \equiv 1 (\text{mod } 2)\}$$
$$\cup \{v_{3i}^j v_{4i}^j : 1 \le i \le m, 1 \le j \le l-1, j \equiv 0 (\text{mod } 2)\} \cup \{u_i^j u_{i+1}^j : 1 \le i \le m-1, 1 \le j \le l, j \equiv 1 (\text{mod } 2)\}$$
$$\cup \{u_1^j u_m^j : 1 \le j \le l, j \equiv 1 (\text{mod } 2)\}.$$

Then $p = |V(CP_1)| = m(3l-2)$ and $q = |E(CP_1)| = 6m(l-1)$. We are defining a bijective function on CP_1 as $\delta : V(CP_1) \to \{1, 2, ..., m(3l-2)\}$ as follows:

For $j \equiv 1 (\text{mod } 2); 1 \le j \le l$

$$\delta(u_i^j) = \begin{cases} \frac{1}{2}(6m(j-1) + i + 1) : 1 \le i \le m; i \equiv 1 (\text{mod } 2); \\ \frac{1}{2}(m(6j-5) + i + 1) : 2 \le i \le m-1; i \equiv 0 (\text{mod } 2). \end{cases}$$

For $j \equiv 0 (\text{mod } 2); 2 \le j \le l-1$

$$\delta(v_{1i}^j) = 3m((j-1) - i + 1) : 1 \le i \le m;$$

$$\delta(v_{2i}^j) = m((3j-2) - i + 1) : 1 \le i \le m;$$

$$\delta(v_{3i}^j) = \begin{cases} \frac{1}{2}(3m(2j-3) + i) : 1 \le i \le m; i \equiv 1 (\text{mod } 2); \\ m(3j-5) + i : 2 \le i \le m-1; i \equiv 0 (\text{mod } 2); \end{cases}$$

$$\delta(v_{4i}^j) = m(3j - 2i + 2) : 1 \le i \le m; i \equiv 1 (\text{mod } 2);$$

$$\delta(v_{5i}^j) = m((3j-1) - i + 1) : 1 \le i \le m;$$

It can be easily followed that with appropriate edge- labels δ refers that the convex polytope CP_1 is $(\frac{m-1}{2},d)$ and $(\frac{m+3}{2},d)$ -arithmetic for d=1.

Theorem 2. The convex polytope CP_2 *is* $(\frac{m-1}{2},d)$ and $(\frac{m+3}{2},d)$ - arithmetic.

Proof. We define the infinite polytope CP_2 with following vertex and edge sets,

$$V(CP_2)=\{u_i^j:1\le i\le m,1\le j\le l, j\equiv 1(\text{mod } 2)\}$$
$$\cup\{v_{1i}^j,v_{2i}^j,v_{3i}^j,v_{4i}^j,v_{5i}^j,v_{6i}^j:1\le i\le m,1\le j\le l-1, j\equiv 0(\text{mod } 2)\}.$$

$$E(CP_2)=\{v_{1i}^ju_i^{j+1},v_{2i}^ju_i^{j+1},v_{3i}^ju_i^{j+1},v_{4i}^ju_i^{j+1},v_{5i}^ju_i^{j+1},v_{6i}^ju_i^{j+1}:1\le i\le m,1\le j\le l-1, j\equiv 0(\text{mod } 2)\}$$
$$\cup\{v_{1i}^{j+1}u_i^j,v_{2i}^{j+1}u_i^j,v_{3i}^{j+1}u_i^j,v_{4i}^{j+1}u_i^j,v_{5i}^{j+1}u_i^j,v_{6i}^{j+1}u_i^j:1\le i\le m,1\le j\le l-2, j\equiv 1(\text{mod } 2)\}$$
$$\cup\{v_{3i}^jv_{4i}^j:1\le i\le m,1\le j\le l-1, j\equiv 0(\text{mod } 2)\}\cup\{u_i^ju_{i+1}^j:1\le i\le m-1,1\le j\le l, j\equiv 1(\text{mod } 2)\}$$
$$\cup\{u_1^ju_m^j:1\le j\le l, j\equiv 1(\text{mod } 2)\}.$$

Then $p=|V(CP_2)|=|V(CP_1)|$ and $q=|E(CP_1)|=|E(CP_2)|$. We are defining a bijective function on CP_2 as $\gamma:V(CP_1)\to\{1,2,...,p\}$ as:

For $j\equiv 1(\text{mod } 2);1\le j\le l$

$$\gamma(u_i^j)=\begin{cases}\frac{1}{2}(7mj-8m+i+1):1\le i\le m; i\equiv 1(\text{mod } 2);\\ \frac{1}{2}(7m(j-1)+i+1):2\le i\le m-1; i\equiv 0(\text{mod } 2).\end{cases}$$

For $j\equiv 0(\text{mod } 2);2\le j\le l-1$

$$\gamma(v_{1i}^j)=\frac{1}{2}(m(7j-8)-2i+2):1\le i\le m;$$

$$\gamma(v_{2i}^j)=\frac{1}{2}(m(7j-6)-2i+2):1\le i\le m;$$

$$\gamma(v_{3i}^j)=\begin{cases}\frac{1}{2}(m(7j-5)+i):1\le i\le m; i\equiv 1(\text{mod } 2);\\ \frac{1}{2}(m(7j-6)+i):2\le i\le m-1; i\equiv 0(\text{mod } 2);\end{cases}$$

$$\gamma(v_{4i}^j)=\frac{1}{2}(7mj-2i+2):1\le i\le m;$$

$$\gamma(v_{5i}^{j}) = \frac{1}{2}(m(7j-4)-2i+2):1 \le i \le m;$$

$$\gamma(v_{6i}^{j}) = \frac{1}{2}(m(7j-2)-2i+2):1 \le i \le m;$$

It can be easily followed that with appropriate edge- labels γ refers that the convex polytope CP_2 is $(\frac{m-1}{2},d)$ and $(\frac{m+3}{2},d)$ -arithmetic for d=1.

Theorem 3. The convex polytope CP_3 *is* $(\frac{m-1}{2},d)$ and $(\frac{m+3}{2},d)$ - arithmetic.

Proof. We are defining the infinite convex polytope CP_3 with following vertex and edge sets,

$$V(CP_3) = \{u_i^j : 1 \le i \le m, 1 \le j \le l, j \equiv 1(\text{mod } 2)\}$$
$$\cup \{v_{1i}^j, v_{2i}^j, v_{3i}^j, v_{4i}^j, v_{5i}^j : 1 \le i \le m, 1 \le j \le l-1, j \equiv 0(\text{mod } 2)\}.$$

$$E(CP_3) = \{x v_{1i}^j u_i^{j+1}, v_{2i}^j u_i^{j+1}, v_{3i}^j u_i^{j+1}, v_{4i}^j u_i^{j+1}, v_{5i}^j u_i^{j+1} : 1 \le i \le m, 1 \le j \le l-1, j \equiv 0(\text{mod } 2)\}$$
$$\cup \{v_{1i}^{j+1} u_i^j, v_{2i}^{j+1} u_i^j, v_{3i}^{j+1} u_i^j, v_{4i}^{j+1} u_i^j, v_{5i}^{j+1} u_i^j : 1 \le i \le m, 1 \le j \le l-2, j \equiv 1(\text{mod } 2)\}$$
$$\cup \{v_{4i}^j v_{3i}^j : 1 \le i \le m-1, 2 \le j \le l-1, j \equiv 0(\text{mod } 2)\} \cup \{u_i^j u_{i+1}^j : 1 \le i \le m-1, 1 \le j \le l, j \equiv 1(\text{mod } 2)\}$$
$$\cup \{u_1^j u_m^j : 1 \le j \le l, j \equiv 1(\text{mod } 2)\} \cup \{v_{41}^j v_{3m}^j :, 2 \le j \le l-1, j \equiv 0(\text{mod } 2)\}.$$

Then $p = |V(CP_3)| = m(3l-2)$ and $q = |E(CP_3)| = 6m(l-1)$. We are defining a bijection on CP_3 as $\vartheta : V(CP_3) \to \{1,2,...,p\}$ as follows:

For $j \equiv 1(\text{mod } 2); 1 \le j \le l$

$$\vartheta(u_i^j) = \delta(u_i^j)$$

For $j \equiv 0(\text{mod } 2); 2 \le j \le l-1$

$$\vartheta(v_{1i}^j) = \delta(v_{1i}^j);$$

$$\vartheta(v_{2i}^j) = \delta(v_{2i}^j);$$

$$\vartheta(v_{3i}^j) = \delta(v_{3i}^j);$$

$$\vartheta(v_{4i}^j) = \delta(v_{4i}^j);$$

$$\vartheta(v_{5i}^j) = \delta(v_{5i}^j).$$

It can be easily followed that with appropriate edge- labels ϑ refers that the convex polytope CP_3 is $(\frac{m-1}{2},d)$ and $(\frac{m+3}{2},d)$ -arithmetic for d=1.

Theorem 4. The convex polytope CP_4 *is* $(\frac{m-1}{2},d)$ and $(\frac{m+3}{2},d)$ - arithmetic.

Proof. We define the infinite polytope CP_4 with following vertex and edge sets,

$$V(CP_4)=\{u_i^j : 1\le i\le m, 1\le j\le l, j\equiv 1(\text{mod } 2)\}$$
$$\cup\{v_{1i}^j, v_{2i}^j, v_{3i}^j, v_{4i}^j, v_{5i}^j, v_{6i}^j : 1\le i\le m, 1\le j\le l-1, j\equiv 0(\text{mod } 2)\}.$$

$$E(CP_4)=\{v_{1i}^j u_i^{j+1}, v_{2i}^j u_i^{j+1}, v_{3i}^j u_i^{j+1}, v_{4i}^j u_i^{j+1}, v_{5i}^j u_i^{j+1}, v_{6i}^j u_i^{j+1} : 1\le i\le m, 1\le j\le l-1, j\equiv 0(\text{mod } 2)\}$$
$$\cup\{v_{1i}^{j+1} u_i^j, v_{2i}^{j+1} u_i^j, v_{3i}^{j+1} u_i^j, v_{4i}^{j+1} u_i^j, v_{5i}^{j+1} u_i^j, v_{6i}^{j+1} u_i^j : 1\le i\le m, 1\le j\le l-2, j\equiv 1(\text{mod } 2)\}$$
$$\cup\{v_{4i}^j v_{5i}^j : 1\le i\le m-1, 2\le j\le l-1, j\equiv 0(\text{mod } 2)\}\cup\{u_i^j u_{i+1}^j : 1\le i\le m-1, 1\le j\le l, j\equiv 1(\text{mod } 2)\}$$
$$\cup\{u_1^j u_m^j : 1\le j\le l, j\equiv 1(\text{mod } 2)\}\cup\{v_{31}^j v_{4m}^j : 2\le j\le l-1, j\equiv 0(\text{mod } 2)\}.$$

Then $p=|V(CP_4)|=|V(CP_2)|$ and $q=|E(CP_4)|=|E(CP_2)|$. We are defining a bijective function on CP_2 as $\eta : V(CP_4)\to\{1,2,\ldots,p\}$ as:

For $j\equiv 1(\text{mod } 2); 1\le j\le l$

$$\eta(u_i^j)=\gamma(u_i^j)$$

For $j\equiv 0(\text{mod } 2); 2\le j\le l-1$

$$\eta(v_{1i}^j)=\gamma(v_{1i}^j);$$

$$\eta(v_{2i}^j)=\gamma(v_{2i}^j);$$

$$\eta(v_{3i}^j)=\gamma(v_{3i}^j);$$

$$\eta(v_{4i}^j)=\gamma(v_{4i}^j);$$

$$\eta(v_{5i}^j)=\gamma(v_{5i}^j);$$

$$\eta(v_{6i}^j)=\gamma(v_{6i}^j);$$

It can be easily followed that with appropriate edge- labels η refers that the convex polytope CP_3 is $(\frac{m-1}{2},d)$ and $(\frac{m+3}{2},d)$ -arithmetic for d=1.

In the end, we are proposing an open problem for researchers.

Open Problem: Can you determine (k, d) arithmetic labeling of the above convex polytopes for other values of d.

Conclusion

By definition, a (p, q) graph G is said to be (k, d)-arithmetic if its vertices can be assigned distinct nonnegative integers so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices, can be arranged in the arithmetic progression $k, k + d, k + 2d.. . . , k + (q - 1)d$. We have shown that the convex polytopes CP_i, for i= 1, 2, 3, 4, as defined, are $(\frac{m-1}{2}, d)$ and $(\frac{m+3}{2}, d)$ -arithmetic.

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