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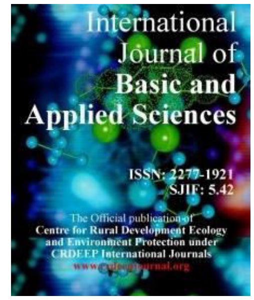

Full Length Research Paper

A Note on Magic Sum of Dijoint Union of Graphs

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ABSTRACT

In 1998, Hikoe Enomoto et al. defined the concept of super edge-magic labelling of graphs which has played very important part in discrete mathematics apart from its many technological, mathematical and chemical applications now a day. The present articles also covers the techniques of obtaining super edge-magic sum mainly of some disjoint union of graphs.

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Introduction

Our main working rests around finite, simple and undirected graphs. We are denoting vertex and edge sets of a graph G by $V(G)$ and $E(G)$ respectively. The (p, q)-graph G is a graph having $|V(G)| = p$ and $|E(G)| = q$. A (p, q)-graph G is said to admit an edge-magic labeling if it admits a bijection $\delta: V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that $\delta(x) + \delta(xy) + \delta(y) = c$ is a constant (called magic sum of G under δ), for each edge $xy \in E(G)$. An edge-magic labeling of G becomes super edge-magic if it has the additional characteristic that $\delta(V(G)) = \{1, 2, \ldots, p\}$. A graph that admits a super edge-magic labeling is said to be super edge-magic. Graph labeling has caught attention of many mathematicians which is not only due to mathematical problems that are being called into question in order to obtain accuracy in this area, but also for its wide application in astronomy, coding, circuit and network designing. Long ago, G. S. Bloom and S. W. Golomb studied applications of graph labeling to various branches of science in their articles, some of their discussions can be seen in [1, 2].

Primarily, H. Enomoto *et al*. [3] gave the idea of super edge-magic labeling of graphs. Figueroa-Centeno *et al.* [6] showed that if G is a super edge-magic bipartite or tripartite graph, then for odd m, mG is super edge-magic. In [3] H. Enomoto *et al.* proved a complete bipartite graph $K_{m,n}$ to be super edge-magic if and only if $m = 1$ or $n = 1$. In [6] it is proved that $K_{1,m} \cup K_{1,n}$ is super edge-magic if either m is a multiple of $n + 1$ or n is a multiple $m + 1$. H. Enomoto *et al.* [3] proved that C_n is super edge-magic if and only if n is odd. $C_3 \cup C_n$ is super edge-magic [9] if and only if $n \geq 6$ and n is even (also see [10]). Graph theorists are still working on this famous conjecture. In [3] H. Enomoto *et al.* showed that the generalized prism $C_{2m+1} \times P_m$ is super edge-magic for every positive integer m (also see [4]). The following G lemma is quite interesting as far as super edge-magic graphs are concerned.

Lemma 1. [4] A (p, q)-graph G is super edge-magic if and only if there exists a bijective function $\delta : V(G) \to \{1, 2, \cdots, p\}$ such that the set $S = \{\delta(x) + \delta(y) | \; xy \in E(G)\}$ consists of q consecutive integers. In such a case, δ extends to a super edge-magic labeling of G with magic constant $c = p + q + s$, where $s = \min(S)$ and $S = \{c - (p + 1), c - (p + 2), \ldots, c - (p + q)\}$.

Kotzig and A. Rosa [11] proved that for every graph G there exists an edge-magic graph H such that $H \cong G \bigcup nK_1$ for some non-negative integer n. This fact leads to the concept of edge-magic deficiency $\mu(G)$ of a graph G, which is the minimum non-negative integer n for which $G \cup nK_1$ is an edge-magic graph. Figueroa- Centeno *et al.* [5] defined a similar concept for super edge-magic graphs. The super edge-magic deficiency of a graph G, denoted by $\mu_s(G)$, is the minimum non-negative integer n such that $G \cup nK_1$ admits a super edge-magic labeling or $+\infty$ if no such integer n *exists*. In [5, 8] Figueroa-Centeno *et al.* have given exact values of the super edge-magic deficiencies of several families of graphs, such as cycles, complete graphs, 2-regular graphs and complete bipartite graphs $K_{2,m}$. They also proved that all forests have finite deficiency. In [13] A. Ngurah, Simanjuntak and Baskoro gave upper bounds for the super edge-magic deficiency of fans, double fans and wheels. In [7] Figueroa-Centeno *et al*. conjectured that every forest with two components has super edge-magic deficiency at most 1.

Concluding the introductory discussion, magic sum of many standard and special families of graphs have been discussed by the graph theorists in the past. Discussion on obtaining magic sum on their disjoint union with isolated vertices is also highly considerable, which is the podium on which this article is erected.

Main Results

We shall prove our main results in the form of the following theorems.

Definition 1. Consider a multicyclic graph H_m, for a cycle C_m, with connection design as follows;

$$V(H_m)=\{x_1^i, x_2^i, x_3^i, x_4^i, y_i, z_i : 1\le i\le m\}$$

$$E(H_m)=\{y_iy_{i+1}, z_iz_{i+1} : 1\le i\le m-1\}\cup\{y_1y_m, z_1z_m\}$$
$$\cup\{y_ix_1^i, y_ix_2^i, y_ix_3^i, y_ix_4^i : 1\le i\le m\}\cup\{z_ix_1^i, z_ix_2^i, z_ix_3^i, z_ix_4^i : 1\le i\le m\}.$$

We use the above definition while proving our results.

Theorem 1. The disjoint union of H_m with $K_{1,m}$ and $(\frac{m-1}{2})K_1$ gives a super edge-magic sum if m is chosen to be odd.

Proof. Consider $H_1\cong H_m\cup K_{1,m}\cup(\frac{m-1}{2})K_1$ with vertex and edge sets:

$$V(H_1)=\{x_1^i, x_2^i, x_3^i, x_4^i : 1\le i\le m\}\cup\{y_i, z_i : 1\le i\le m\}$$
$$\cup\{k_i : 1\le i\le m\}\{d_i : 1\le i\le \frac{m-1}{2}\}\cup\{k\}.$$

$$E(H_1)=\{y_iy_{i+1}, z_iz_{i+1} : 1\le i\le m-1\}\cup\{y_1y_m, z_1z_m\}$$
$$\cup\{y_ix_1^i, y_ix_2^i, y_ix_3^i, y_ix_4^i : 1\le i\le m\}\cup\{z_ix_1^i, z_ix_2^i, z_ix_3^i, z_ix_4^i : 1\le i\le m\}$$
$$\cup\{kk_i : 1\le i\le m\}.$$

Then $p=|V(H_1)|=\frac{15m+1}{2}$ and $q=|E(H_1)|=11m$. We are defining a bijection as

$$f : V(H_1)\to\{1,2,...,\frac{15m+1}{2}\}\text{ as follows:}$$

$$f(x_1^i)=4m-i+1 : 1\le i\le m.$$

$$f(x_2^i)=\begin{cases}\frac{1}{2}(5m+i) : 1\le i\le m; i\equiv 1(\mathrm{mod}\ 2);\\ \frac{1}{2}(4m+i) : 2\le i\le m-1; i\equiv 0(\mathrm{mod}\ 2).\end{cases}$$

$$f(x_3^i)=6m-i+1 : 1\le i\le m.$$

$$f(x_4^i)=5m-i+1 : 1\le i\le m.$$

$$f(y_i)=\begin{cases}\frac{1}{2}(i+2m+1) : 1\le i\le m; i\equiv 1(\mathrm{mod}\ 2);\\ \frac{1}{2}(i+3m+1) : 2\le i\le m-1; i\equiv 0(\mathrm{mod}\ 2).\end{cases}$$

$$f(z_i)=\begin{cases}\frac{1}{2}(i+12m+1) : 1\le i\le m; i\equiv 1(\mathrm{mod}\ 2);\\ \frac{1}{2}(i+13m+1) : 2\le i\le m-1; i\equiv 0(\mathrm{mod}\ 2).\end{cases}$$

$$f(k_i)=i : 1\le i\le m.$$

$$f(k)=\frac{15m+1}{2};$$

$$f(d_i)=7m+i : 1\le i\le \frac{m-1}{2}.$$

The edge-sums generated by the above scheme form a sequence of consecutive integers $\frac{5m+3}{2}, \frac{5m+5}{2}, ..., \frac{27m+1}{2}$. Therefore by Lemma 1, f extends to a super edge-magic labeling of H_1 with magic sum $21m+2$.

We propose the following open problem for further work here.

Open Problem 1. *For even m, can you find a magic- sum for the above defined graph* H_1?

Let us move to our second result.

Theorem 2. The disjoint union of H_m with mP_2 gives a super edge-magic sum if m is chosen to be odd.

Proof. Consider a graph $H_2\cong H_m\cup mP_2$ for odd m with following vertex and edge connection:

$$V(H_2)=\{x_1^i, x_2^i, x_3^i, x_{24}^i : 1\le i\le m\}\cup\{y_i, z_i : 1\le i\le m\}$$
$$\cup\{p_i, q_i : 1\le i\le m\}.$$

$$E(H_2)=\{y_iy_{i+1}, z_iz_{i+1} : 1\le i\le m-1\}\cup\{y_1y_m, z_1z_m\}$$
$$\cup\{y_ix_1^i, y_ix_2^i, y_ix_3^i, y_ix_4^i : 1\le i\le m\}\cup\{z_ix_1^i, z_ix_2^i, z_ix_3^i, z_ix_4^i : 1\le i\le m\}$$
$$\cup\{p_iq_i : 1\le i\le m\}.$$

Then $p=|V(H_2)|=8m$ and $q=|E(H_2)|=11m$. Now, we are defining a labeling

$g: V(H_2) \to \{1,2,\ldots,8m\}$ as:

$$g(x_1^i) = 4m - i + 1 : 1 \le i \le m.$$

$$g(x_2^i) = \begin{cases} \frac{1}{2}(5m+i) : 1 \le i \le m; i \equiv 1 (\mathrm{mod}\ 2); \\ \frac{1}{2}(4m+i) : 2 \le i \le m-1; i \equiv 0 (\mathrm{mod}\ 2). \end{cases}$$

$$g(x_3^i) = 6m - i + 1 : 1 \le i \le m.$$

$$g(x_4^i) = 5m - i + 1 : 1 \le i \le m.$$

$$g(y_i) = \begin{cases} \frac{1}{2}(i+2m+1) : 1 \le i \le m; i \equiv 1 (\mathrm{mod}\ 2); \\ \frac{1}{2}(i+3m+1) : 2 \le i \le m-1; i \equiv 0 (\mathrm{mod}\ 2). \end{cases}$$

$$g(z_i) = \begin{cases} \frac{1}{2}(i+12m+1) : 1 \le i \le m; i \equiv 1 (\mathrm{mod}\ 2); \\ \frac{1}{2}(i+13m+1) : 2 \le i \le m-1; i \equiv 0 (\mathrm{mod}\ 2). \end{cases}$$

$$g(q_i) = \begin{cases} \frac{1}{2}(15m-i+2) : 1 \le i \le m; i \equiv 1 (\mathrm{mod}\ 2); \\ \frac{1}{2}(16m-i+2) : 2 \le i \le m-1; i \equiv 0 (\mathrm{mod}\ 2). \end{cases}$$

$$g(p_i) = i : 1 \le i \le m.$$

The set of all edge-sums generated by the bijection g forms a consecutive integer sequence $\frac{5m+3}{2}, \frac{5m+5}{2}, \ldots, \frac{27m+1}{2}$. Therefore by Lemma 1, g extends to a super edge-magic labeling of H_2 with magic sum, same as obtained in Theorem 1.

Open Problem 2. *For even m, can you find a magic- sum for the graph* H_2?

Theorem 3. The disjoint union of H_m with P_{m+1} gives a super edge-magic sum if m is chosen to be odd.

Proof. Consider $H_3 \cong H_m \cup P_{m+1}$ for odd m with following vertex and edge connection:

$$V(H_3) = \{x_1^i, x_2^i, x_3^i, x_{24}^i : 1 \le i \le m\} \cup \{y_i, z_i : 1 \le i \le m\}$$
$$\cup \{p_i : 1 \le i \le m+1\}.$$

$$E(H_3) = \{y_i y_{i+1}, z_i z_{i+1} : 1 \le i \le m-1\} \cup \{y_1 y_m, z_1 z_m\}$$
$$\cup \{y_i x_1^i, y_i x_2^i, y_i x_3^i, y_i x_4^i : 1 \le i \le m\} \cup \{z_i x_1^i, z_i x_2^i, z_i x_3^i, z_i x_4^i : 1 \le i \le m\}$$
$$\cup \{p_i p_{i+1} : 1 \le i \le m\}.$$

Then $p = |V(H_3)| = 7m+1$ and $q = |E(H_3)| = 11m$. Now, we are defining a labeling

$h: V(H_3) \to \{1,2,\ldots,7m+1\}$ as:

$$h(x_1^i) = 7m - 2i + 1 : 1 \le i \le m.$$

$$h(x_2^i) = \begin{cases} \frac{1}{2}(3m+i+1) : 1 \le i \le m; i \equiv 1 (\mathrm{mod}\ 2); \\ \frac{1}{2}(4m+i+1) : 2 \le i \le m-1; i \equiv 0 (\mathrm{mod}\ 2). \end{cases}$$

$$h(x_3^i) = 11m - 2i + 3 : 1 \le i \le m.$$

$$h(x_4^i) = 9m - 2i + 3 : 1 \le i \le m.$$

$$h(y_i) = \begin{cases} \frac{1}{2}(i+m+2) : 1 \le i \le m; i \equiv 1 (\mathrm{mod}\ 2); \\ \frac{1}{2}(i+2m+2) : 2 \le i \le m-1; i \equiv 0 (\mathrm{mod}\ 2). \end{cases}$$

$$h(z_i) = \begin{cases} \frac{1}{2}(i+11m+2) : 1 \le i \le m; i \equiv 1 (\mathrm{mod}\ 2); \\ \frac{1}{2}(i+12m+2) : 2 \le i \le m-1; i \equiv 0 (\mathrm{mod}\ 2). \end{cases}$$

$$h(p_i) = \begin{cases} \frac{1}{2}(i+1) : 1 \le i \le m; i \equiv 1 (\mathrm{mod}\ 2); \\ \frac{1}{2}(13m+i+1) : 2 \le i \le m+1; i \equiv 0 (\mathrm{mod}\ 2). \end{cases}$$

The edge-sums generated by the bijection h forms a consecutive integer sequence $\frac{3m+5}{2}, \frac{3m+7}{2}, \ldots, \frac{25m+3}{2}$. Therefore by Lemma 1, h extends to a super edge-magic labeling of H_3 with magic sum $\frac{39m+7}{2}$.

Open Problem 3. *For even m, can you find a magic- sum for the graph* $H_m \cup P_{m+1}$?

Conclusion

In this article, we have mainly focused on computing the super edge- magic sum of disjoint union of graphs, in particular of $H_m \cup K_{1,m} \cup (\frac{m-1}{2})K_1$, $H_m \cup mP_2$ and $H_m \cup P_{m+1}$. We have shown these combinations have constant super edge-magic sums for their parities present. We have also proposed three open problems for further working in this field in order to obtain further accuracy.

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