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Oscillation of a Class of Fifth Order Neutral Differential Equation

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ABSTRACT

The aim of this paper is to derive the oscillation criteria of fifth order neutral differential equation of the form

$$\left[r(t)\left(m(t)y(t)+p(t)y(\tau(t))\right)''''\right]'+f(t)G\left(y(\sigma(t))\right)=0\ ;\quad t\ge t_0>0.$$

By applying Riccati Transformation technique sufficient conditions for the oscillation of this equation is obtained.

1. Introduction

In this paper, we study the oscillation of all solutions of the fifth order neutral differential equation

$$\left[r(t)\left(m(t)y(t)+p(t)y(\tau(t))\right)''''\right]'+f(t)G\left(y(\sigma(t))\right)=0\ ;\quad t\ge t_0>0. \qquad (1)$$

Differential equations have received a lot of attention, and it is an active research area among engineers and scientists. Several fields of science, including Biology, Chemistry and medicine use delay differential equations. A neutral differential equation is a differential equation in which the highest order derivative of the unknown function appears with the argument t and one or more delayed arguments. Neutral differential equations are used in numerous applications in Natural science and technology. In recent years, there has been an increasing interest on the oscillatory behavior of second order nonlinear or quasilinear delay differential equations with impulse action. We refer to the papers [13, 18].

A great deal of work has been done on various aspects of differential equations of third order see for example [14,15,20]. In [16], Osama Moaaz, Rami Ahmad El-Nabulsi, Ali Muhib, Sayed K. Elagan and Mohammed Zakarya, studied the new oscillation criteria for fourth order differential equation of the form

$$\left(r(t)\left(\left(x(t)+p(t)x(\delta(t))\right)'''\right)^{\alpha}\right)'+q(t)x^{\beta}(\emptyset(t))=0$$

By using the Riccati technique new criteria for oscillation is obtained without requiring the existence of the unknown function.

In [5] Dassios and Bazighifan studied the oscillatory behaviour of nonlinear differential equations

$$\left(r(s)\left(y'''(s)\right)^{\gamma}\right)'+q(s)x^{\gamma}(\sigma(s))=0, \qquad (2)$$

Where $\gamma=\delta$, and by using the Riccati technique, they proved that the solutions to (2) are oscillatory or converge to zero as $s\to\infty$.

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In [10] John R. Graef, Hakan Avcı, Osman Ozdemir and Ercan Tun¸c, studied the Oscillatory behavior of a fifth-order differential equation of the form

$$y^5(t) + q(t)x(\sigma(t) = 0$$

with unbounded neutral coefficients, where $t \geq t_0 > 0$, where $y(t) = x(t) + p(t)x(\tau(t)$ and obtained the results by a comparison with first-order delay differential equations whose oscillatory characters are known.

In recent years, there has been a great deal of interest in studying the oscillatory behavior of solutions of various types of differential equations; see [2,3,4,6,7,9,12,13,18,19,21,22] and the monograph [1].In the investigation of oscillation of solutions of differential equations Riccati type transformations are useful. By using Riccati type Transformations, we establish some new sufficient conditions for the oscillation of solutions of equation (1).

By a solution of equation (1), we mean a function $x(t) \in C'\left(\left[T_y, \infty\right)\right) T_y \geq t_0$,which has the property $r(t)(m(t)y(t) + p(t)y(\tau(t))'''' \in C'([T_y, \infty), \mathbb{R})$ and satisfies equation (1.1) on $[T_y, \infty)$. We consider only those solutions y of equation (1) which satisfy $sup\{|y(t): t \geq T|\} > 0$ for all $T \geq T_y$, and assume that the equation (1) possess such solution.

A solution of equation (1) is called oscillatory if it has arbitrary large zeros on $[T_y, \infty)$; otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions oscillate. Unless otherwise stated, when we write a functional inequality, it will be assumed to hold for sufficiently large t in our subsequent discussion.

2. Method

By using the Generalized Riccati transformation technique and few Lemmas, sufficient conditions for the oscillation of equation (1) are obtained under condition

$$R(t) = \int_{t_0}^{\infty} \frac{1}{r(s)} ds < \infty.$$

3. Main Results

We need the following in our discussion

$[H_1]$: $r(t), m(t) \epsilon C'[t_0, \infty), r'(t) \geq 0, r(t) > 0,$

$[H_2]$: $f(t), \sigma, \tau \epsilon C[t_0, \infty), f(t) \geq 0,\ G \epsilon C(R, R), r(t), m(t) \epsilon C'[t_0, \infty), r'(t) \geq 0, r(t) > 0,$

and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$.

$[H_3]$: $\frac{G(y(\sigma(t))}{y(\sigma(t)} \geq k$

$[H_4]$: $\int_{t_0}^{\infty} \frac{1}{r(s)} ds < \infty.$

Here $x(t) = m(t)y(t) + p(t)y(\tau(t))$

In this for convenience, we denote

$$R(t) = \int_{t_0}^{\infty} \frac{1}{r(s)} ds < \infty$$

$$x(t) = m(t)y(t) + p(t)y(\tau(t)) \tag{3}$$

$$m(t)y(t) = x(t) - p(t)y(\tau(t))$$

$$y(t) = \frac{1}{m(t)}\{x(t) - p(t)y(\tau(t))\}$$

$$y(t) = \frac{1}{m(t)}\{x(t) - p(t)x(\tau(t))\}$$

$$y(t) = \frac{1}{m(t)}\{1 - p(t)\}x(t)$$

$$y(\sigma(t)) = \frac{1}{m(\sigma(t))}\{1 - p(\sigma(t))\}x(\sigma(t)) \tag{4}$$

Lemma 3.1: Assume that $x \in C^n(I, \mathbb{R})$ such that $x(t) > 0$ for all $t \geq a$ and $x^n(t)$ is nonpositive for $t \geq a$ and does not vanish identically on any $[T, \infty) \subseteq I$ where $I = [a, \infty)$.If n is even (or odd), then there exists $l \in \{1,3, \dots \dots \dots n-1\}$(resp; $l \in \{0,2, \dots \dots n-1\}$) such that for all sufficiently large t,

$x(t)x^j(t) > 0\ for\ j = 0,1, \dots . . l, and\ (-1)^{n+j-1}x(t)x(t)^j > 0\ for\ j = l+1, l+2, \dots . n-1.$ Furthermore,

$$|x'(\sigma(t))| \geq \frac{\sigma^{l-1}(t)\left(t - \sigma(t)\right)^{n-l-1}}{2^{l-1}(l-1)!\,(n-1-l)!} |x^{(n-1)}(t)| \tag{5}$$

For all sufficiently large t, where $\sigma \in C'(I,\mathbb{R})$ satisfies $0 < \sigma(t) < t$, $\sigma'(t) \geq 0$ and $\lim_{t\to\infty} \sigma(t) = \infty$.

Proof: The existence of l in the above result is due to Kiguradze and is well known. (see, e.g. [11]).

Next, we will prove the inequality (5) for even n (the odd case being similar) as follows (cf. [17]). First, we may assume that there is a $T > 0$ such that

$x(t) > 0, x'(t) > 0, \ldots \ldots x^{l}(t) > 0, (-1)^{n+1}x^{(l+1)}(t) > 0, \ldots \ldots (-1)^{2n-2}x^{(n-1)}(t) > 0$ for $t \geq T$. By Taylor's formula, we have

$$x'(s) = x'(T) + x''(T)(s-T) + \cdots \frac{x^{(l)(s^*)}}{(l-1)!}(s-T)^{l-1} \geq \frac{x^{(l)(s^*)}}{(l-1)!}(s-T)^{l-1}, \qquad (6)$$

where $T \leq s^* \leq s$. Furthermore,

$$x'(s) \geq \frac{x^{(l)}(s)}{(l-1)!}(s-T)^{l-1}, \quad s \geq T, \qquad (7)$$

since $x^{(l+1)}(t) < 0$ for $t \in [T,\infty)$. Next, we pick $T_1 \geq T$ such that $1 - \frac{T}{\sigma(t)} > \frac{1}{2}$ for $t \geq T_1$.

Then $\sigma(t) > 2T$ and

$$x'(\sigma(t)) \geq \frac{x^{l}(\sigma(t))}{(l-1)!}\sigma^{l-1}(t)\left(1 - \frac{T}{\sigma(t)}\right)^{l-1} \geq \frac{\sigma^{l-1}(t)}{2^{l-1}(l-1)!}x^{l}(\sigma(t)), \quad t \geq T_1 (8)$$

By Taylor's formulae again, we get

$$x^{(l)}(\sigma(t)) = x^{(l)}(t) + (-1)x^{(l+1)}(t)(t-\sigma(t)) + \cdots + (-1)^{n-l-1}\frac{x^{(n-1)}(t^*)}{(n-l-1)!}(t-\sigma(t))^{n-l-1}, \qquad (9)$$

Hence $\sigma(t) \leq t^* \leq t$. Hence

$$x^{(l)}(\sigma(t)) \geq \frac{x^{(n-1)}(t^*)}{(n-l-1)!}(t-\sigma(t))^{n-l-1} \geq \frac{x^{(n-1)}(t)}{(n-l-1)!}(t-\sigma(t))^{n-l-1} (10).$$

Combining (10) and (8) we see that

$$x'(\sigma(t)) \geq \frac{\sigma^{l-1}(t)}{2^{l-1}(l-1)!(n-1-l)!}x^{(n-1)}(t)(t-\sigma(t))^{n-l-1} (11)$$

Lemma 3.2: Let the conditions $(H_1)-(H_4)$ be satisfied and assume that y is an eventually positive solution of equation (1). Then there exists $t_1 \in [t_0,\infty)$ such that the corresponding function $x(t)$ satisfies one of the following two cases

(i) $x(t) > 0,\ x'(t) > 0,\ x''(t) > 0,\ x'''(t) > 0,\ x''''(t) > 0,$ *and* $x'''''(t) \leq 0,$

(ii) $x(t) > 0,\ x'(t) > 0,\ x''(t) > 0,\ x'''(t) < 0,\ x''''(t) > 0,$ *and* $x'''''(t) \leq 0,$

for $t \geq t_0$.

Theorem3.1: Assume $(H_1)-(H_4)$ and **Lemma 3.1** hold and there exists a positive nondecreasing function $\rho \in C'([t_0,\infty),\mathbb{R})$ such that

$$\limsup_{t\to\infty} \int_{t_0}^{t}\left\{k\rho(s)f(s)\frac{1}{m(\sigma(s))}\{1-p(\sigma(s)\} - \frac{(l-1)!(4-l)!\left(\rho'(s)\right)^2}{2^{4-l}\sigma^{l-1}(s)(s-\sigma(s))^{4-l}\sigma'(s)\rho(s)}\right\}ds = \infty, \qquad (12)$$

Here $l = 1$ *and* $l = 4$ for every $t \geq t_0$. Then every solution of equation (1) is oscillatory.

Proof: Let $y(t)$ be a solution of (1). Suppose to the contrary that $y(t)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\sigma(t)) > 0$ for all $t \geq t_0$. From equation (1) and by **Lemma 3.1**, we may suppose that $y''''(t) > 0$ hold for $t \geq t_0$.

Define

$$\omega(t) = \rho(t)\frac{r(t)\left(x''''(t)\right)}{x(\sigma(t))} (13)$$

$$\omega'(t) = \rho'(t)\frac{r(t)\left(x''''(t)\right)}{x(\sigma(t))} + \rho(t)\left\{\frac{r(t)\left(x''''(t)\right)}{x(\sigma(t))}\right\}'$$

$$= \rho'(t)\frac{r(t)\left(x''''(t)\right)}{x(\sigma(t))} + \rho(t)\left\{\frac{x(\sigma(t)\left[r(t)\left(x''''(t)\right)\right]' - r(t)\left(x''''(t)\right)x'(\sigma(t))\sigma'(t)}{x^2(\sigma(t))}\right\}$$

$$= \rho'(t)\frac{r(t)\left(x'''(t)\right)}{x(\sigma(t))} + \rho(t)\left\{\frac{\left[r(t)\left(x'''(t)\right)\right]'}{x(\sigma(t))}\right\} - \rho(t)\left\{\frac{r(t)\left(x'''(t)\right)x'(\sigma(t))\sigma'(t)}{x^2(\sigma(t))}\right\}$$

From (13) and (1) we obtain,

$$= \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)\left\{\frac{f(t)G(y(\sigma(t)))}{x(\sigma(t))}\right\} - \rho(t)\left\{\frac{r(t)\left(x'''(t)\right)x'(\sigma(t))\sigma'(t)}{x^2(\sigma(t))}\right\}$$

Since $\frac{G(y(\sigma(t)))}{y(\sigma(t))} \geq k$ we obtain,

$$\leq \frac{\rho'(t)}{\rho(t)}\omega(t) - \left\{\frac{k\rho(t)f(t)y(\sigma(t))}{x(\sigma(t))}\right\} - \rho(t)\left\{\frac{r(t)\left(x'''(t)\right)x'(\sigma(t))\sigma'(t)}{x^2(\sigma(t))}\right\}$$

Substituting (4) in the above inequality we have,

$$\leq \frac{\rho'(t)}{\rho(t)}\omega(t) - \left\{k\rho(t)f(t)\frac{1}{m(\sigma(t))}\{1-p(\sigma(t)\}\right\} - \rho(t)\left\{\frac{r(t)\left(x'''(t)\right)x'(\sigma(t))\sigma'(t)}{x^2(\sigma(t))}\right\}$$

Using the inequality (5) in the above inequality and simplifying we obtain,

$$\leq -\left\{k\rho(t)f(t)\frac{1}{m(\sigma(t))}\{1-p(\sigma(t)\}\right\} + \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\sigma^{l-1}(t)\left(t-\sigma(t)\right)^{4-l}\sigma'(t)}{2^{l-1}(l-1)!\,(4-l)!\,\rho(t)}\omega^2(t) \quad (14)$$

$$\leq -\left\{k\rho(t)f(t)\frac{1}{m(\sigma(t))}\{1-p(\sigma(t)\}\right\} - \frac{\sigma^{l-1}(t)\left(t-\sigma(t)\right)^{4-l}\sigma'(t)}{2^{l-1}(l-1)!\,(4-l)!\,\rho(t)}\left(\omega - \frac{2^{l-1}(l-1)!\,(4-l)!\,\rho'(t)}{2\,\sigma^{l-1}(t)\left(t-\sigma(t)\right)^{4-l}\sigma'(t)}\right)^2$$
$$+ \frac{(l-1)!\,(4-l)!\left(\rho'(t)\right)^2}{2^{4-i}\,\sigma^{l-1}(t)\left(t-\sigma(t)\right)^{4-l}\sigma'(t)\rho(t)}$$

$$\leq -\left\{k\rho(t)f(t)\frac{1}{m(\sigma(t))}\{1-p(\sigma(t)\} - \frac{(l-1)!\,(4-l)!\left(\rho'(t)\right)^2}{2^{4-i}\,\sigma^{l-1}(t)\left(t-\sigma(t)\right)^{4-l}\sigma'(t)\rho(t)}\right\}, \quad t \geq t_0.$$

Hence

$$\int_{t_0}^{t}\left\{k\rho(s)f(s)\frac{1}{m(\sigma(s))}\{1-p(\sigma(s)\} - \frac{(l-1)!\,(4-l)!\left(\rho'(s)\right)^2}{2^{4-l}\,\sigma^{l-1}(s)\left(s-\sigma(s)\right)^{4-l}\sigma'(s)\rho(s)}\right\}ds \leq -\int_{t_0}^{t}\omega'(s)ds = \omega(t_0) - \omega(t) \leq \omega(t_0) \quad (15)$$

for all $t \geq t_0$, which is a contradiction to our assumption (12).

Theorem 3.2: Let $(H_1) - (H_4)$ hold. If there exists a real valued nondecreasing differentiable function $\rho(t)$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\left\{k\rho(s)f(s)\frac{1}{m(\sigma(s))}\{1-p(\sigma(s)\}\right\} - \frac{1}{4}\frac{\left(\rho'(s)\right)^2}{\rho(s)}\frac{2^{l-1}(l-1)!\,(4-l)!}{\sigma^{l-1}(s)\left(s-\sigma(s)\right)^{4-l}\sigma'(s)}\right)ds = \infty \quad (16)$$

Then every solution of equation (1) is oscillatory

Proof: Let $x(t)$ be a positive solution of equation (1). In this case we define again the function $\omega(t)$ by (13) and proceeding as in the proof of **Theorem 3.1**, to obtain (14).

Applying the inequality

$$Au - Bu^2 \leq \frac{1}{4}\frac{A^2}{B}$$

with

$$A = \frac{\rho'(t)}{\rho(t)}, \qquad B = \frac{\sigma^{l-1}(t)\left(t-\sigma(t)\right)^{4-l}\sigma'(t)}{2^{l-1}(l-1)!\,(4-l)!\,\rho(t)} \text{ and } \quad u = \omega(t).$$

in equation (14)) we get,

$$\omega'(t) \leq -\left\{k\rho(t)f(t)\frac{1}{m(\sigma(t))}\{1-p(\sigma(t)\}\right\} + \frac{1}{4}\frac{\left(\rho'(t)\right)^2}{\rho(t)}\frac{2^{l-1}(l-1)!\,(4-l)!}{\sigma^{l-1}(t)\left(t-\sigma(t)\right)^{4-l}\sigma'(t)}$$

Integrating the resultant inequality from t_0 to t, we obtain

$$\int_{t_0}^{t}\left(\left\{k\rho(s)f(s)\frac{1}{m(\sigma(s))}\{1-p(\sigma(s)\}\right\} - \frac{1}{4}\frac{\left(\rho'(s)\right)^2}{\rho(s)}\frac{2^{l-1}(l-1)!\,(4-l)!}{\sigma^{l-1}(s)\left(s-\sigma(s)\right)^{4-l}\sigma'(s)}\right)ds < \omega(t_0) \quad (18)$$

which contradicts (16) as $t \to \infty$. This completes the proof.

4. Conclusion

In this paper some oscillatory criteria for a fifth order differential equation has been established by the use of the Riccati transformation technique. The process can be extended to other types of high-order differential equations as well as fractional differential equations, such as fifth-order fractional differential or difference equations with a neutral term, damping term and so on, which will be helpful for further research. Also, this work highlights the relevance of the theory of fifth-order differential equations to various fields of mathematics and practical sciences, emphasizing the importance of continued research in this area.

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