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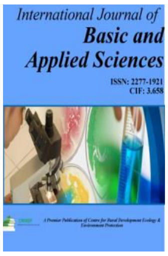

Research Paper

On the Oscillatory Behaviour of Fourth Order Differential Equations with a sublinear neutral term

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ABSTRACT

The aim of this paper is to derive the oscillation properties of fourth order Emden-Fowler differential equation with a sublinear neutral term of the form

$$\left(a(t)\left(\left(m(t)x(t)+p(t)x^{\alpha}(\tau(t))\right)'''\right)^{\gamma}\right)'+q(t)x^{\beta}\big(\sigma(t)\big)=e(t), \qquad t\ge t_0, \qquad (1)$$

is considered. Some new sufficient conditions for oscillation of all solutions of a class of second order differential equations with sublinear neutral terms are given under the condition that $\int_{t_0}^{\infty}\frac{1}{a^{\frac{1}{\gamma}}(s)}ds=\infty$. *By applying the Riccati Transformation technique sufficient conditions for the oscillation of the equation is obtained. Also, the results are an extension and simplification as well as improvement of the previous results.*

1. Introduction

In this paper, we study the oscillatory behavior of fourth order Emden-Fowler differential equations with sublinear neutral term of the form

$$\left(a(t)\left(\left(m(t)x(t)+p(t)x^{\alpha}(\tau(t))\right)'''\right)^{\gamma}\right)'+q(t)x^{\beta}\big(\sigma(t)\big)=e(t), \qquad t\ge t_0.$$

In last few years there has been much research activity concerning oscillatory behavior of various classes of differential equations. Oscillatory phenomena arise in various models from real world applications see e.g. [3,8,10].

Oscillatory behaviour of second order differential equations is extensively studied by [4,6,7,9,13,14] and also Oscillatory behaviour of Third order differential equations is extensively studied by [16,17], and for the Oscillatory behavior of fourth order we refer to [11,12,15].

Hui Li et al. [5] studied the second order Emden- Fowler neutral delay differential equations of the form

$$\left(a(t)\left(z'(t)\right)^{\alpha}\right)'+f(t)x^{\beta}\big(\tau(t)\big)=0$$

where $z(t)=x(t)+p(t)x((\sigma))$ and obtained oscillation criteria by applying the inequality Technique and Riccati Transformation.

Agarwal et al. [2] considered the second order Emden-Fowler differential equations of the form

$$\left(a(t)\left(\left(y(t)+p(t)y(\sigma(t))'\right)^{\alpha}\right)'+f(t)y^{\gamma}\big(\tau(t)\big)=0, \quad t\ge t_0$$

A.A El-Gabera et al. [4] studied the oscillatory behavior of second order differential equations of the form

$$\left(r(t)\left(z'(t)\right)^{\gamma}\right)'+f(t)\big(z'(t)\big)^{\gamma}+h\big(t,y(\varphi(t)\big)=0, \quad t\ge t_0>0$$

and established some new sufficient conditions.

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By a solution of equation (1) we mean a function $x(t) \in C([T_x,\infty))$, $T_x \geq t_0$ which has the properties $z(t) \in C'([T_x,\infty)), a(t)\left(z'''(t)\right)^{\gamma} \in C'(([T_x,\infty))$ and satisfies equation (1) on $([T_x,\infty))$.We consider only those solutions x of equation (1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$, and assume that the equation (1) possesses such solutions. As usual, a solution of equation (1) is called oscillatory if it has a zero on $[T,\infty)$ for all $T \geq T_x$; otherwise it is called nonoscillatory. If all solutions of a differential equations are oscillatory, then the equation itself is called oscillatory

2. Method

In this paper we use few Lemmas and Inequality (16) which are helpful to prove our results by applying the Riccati Transformation technique.

3. Main Results

We need the following in our discussion

(H_1): $0 < \alpha \leq 1$, β and γ are ratios of odd natural numbers, $m(t)$ is a real valued continuous function.

(H_2): $a \in C'[t_0,\infty),(0,\infty))$, $a'(t) \geq 0$, $p,q \in C[t_0,\infty),(0,\infty))$, $\lim_{t\to\infty} p(t) = 0$ and $q(t) > 0$ $e(t) \geq 0$, $\tau(t) \leq t$, $\tau'(t) \geq 0$ and $\sigma(t) \leq t, \sigma'(t) > 0$, $\lim_{t\to\infty} \tau(t) = \infty$.

(H_3): We define
$$R(t) = \int_{t_0}^{\infty} \frac{1}{a^{\frac{1}{\gamma}}(s)} ds = \infty. \quad (2)$$

Lemma 3.1([1]). Suppose that $\sigma \in C^n([v_0,\mathbb{R}^+)$, where $\rho^{(n)}(v)$ has a constant sign and is non-zero on $[v_0,\infty)$. Additionally, suppose that there is $v_1 \geq v_0$ such that $\rho^{n-1)}(v)\rho^n(v) \leq 0$ for every $v \geq v_1$. If $\lim_{v\to\infty} \rho(v) \neq 0$, then for any $\delta \in (0,1)$, there is $v_\epsilon \in [v_1,\infty)$ such that $\rho(v) \geq \frac{\epsilon}{(n-1)!} v^{n-1}|\rho^{(n-1)}(v)|$, for $\rho \in [v_\epsilon,\infty)$.

Lemma 3.2. Let $\rho \in C^n([v_0,\infty),(0,\infty))$, $\rho^{(i)}(v) > 0$ for $i = 1,2,\ldots..n$, and $\rho^{(n+1)}(v) \leq 0$, eventually. Then, eventually, $\frac{\rho(v)}{\rho'(v)} \geq \frac{\epsilon v}{n}$ for every $\epsilon \in (0,1)$.

Lemma 3.3. Assume that $x(t)$ is an eventually positive solution of (1). Then, $x(t)$ eventually satisfies the following cases

C_1: $z(t) > 0, z'(t) > 0, z''(t) > 0, z'''(t) > 0, (a(t)(z'''(t))^{\gamma})' < 0,$

C_2: $z(t) > 0, z'(t) > 0, z''(t) < 0, z'''(t) > 0, (a(t)(z'''(t))^{\gamma})' < 0,$

Theorem 3.1: Assume that (2) holds. If $\beta \geq \gamma$ and there is a nondecreasing function $\rho \in C'([t_0,\infty),(0,\infty))$ such that
$$\lim_{t\to\infty} sup \int_{t_0}^{t} \left(\rho(s) \left(q(s) \frac{1}{m^\beta(\sigma(s))} \left(1 - \frac{p(\sigma(s))}{c_1{}^{1-\alpha}}\right)^\beta - \frac{e(s)}{c_2^\beta} \right) - \frac{2^\gamma a(\sigma(s))(\rho'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1} c_2^{\frac{\beta-\gamma}{\gamma}} (\epsilon\sigma^2(s)\sigma'(s)\rho(s))^\gamma} \right) ds = \infty. \quad (3)$$

holds for every $c_1, c_2 > 0$, then (1) is oscillatory.

Proof: We assume for contradiction that (1) has an eventually positive solution of $x(t)$.

Set
$$z(t) = m(t)x(t) + p(t)x^\alpha(\tau(t)) \quad (4)$$

Then $z(t) \geq x(t)$. By (1) and (2), we obtain that for $t_1 \geq t_0$
$$z(t) > 0,\ z'(t) > 0,\ z''(t) > 0, z'''(t) > 0,\ (a(t)(z'''(t))^\gamma)' \leq 0, \quad t \geq t_1. \quad (5)$$

Since $\sigma(t) \leq t$, then we have from (5) that
$$a(t)\left(z'''(t)\right)^\gamma \leq a(\sigma(t))\left(z'''(\sigma(t))\right)^\gamma, \quad t \geq t_1 \quad (6)$$

Since that $z'(t) > 0$. Hence there exists a constant $c_1 > 0$ such that $z(t) \geq c_1$ for all t large enough.

By (5), one gets
$$m(t)x(t) \geq z(t) - p(t)z^\alpha(\tau(t))$$
$$x(t) \geq \frac{1}{m(t)}\left(1 - \frac{p(t)}{c_1{}^{1-\alpha}}\right) z(t) \quad (7)$$

Then from equation (1), we have
$$(a(t)((m(t)x(t) + p(t)x^\alpha(\tau(t))''')^\gamma)' + q(t)\frac{1}{m^\beta(\sigma(t))}\left(1 - \frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta z^\beta(\sigma(t)) \leq e(t) \quad (8)$$

Define
$$\omega(t) = \rho(t)\frac{a(t)\left(z'''(t)\right)^\gamma}{z^\beta(\sigma(t))}, \quad t \geq t_1 \quad (9)$$

$$\omega'(t) = \rho'(t)\frac{a(t)\left(z'''(t)\right)^{\gamma}}{z^{\beta}\left(\sigma(t)\right)} + \rho(t)\frac{\left(a(t)\left(z'''(t)\right)^{\gamma}\right)'}{z^{\beta}\left(\sigma(t)\right)} - \beta\rho(t)\frac{a(t)\left(z'''(t)\right)^{\gamma} z'\left(\sigma(t)\right)\sigma'(t)}{z^{\beta+1}\left(\sigma(t)\right)} \tag{10}$$

We see from (7), (8), (9) and (10) we obtain

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)\left(q(t)\frac{1}{m^{\beta}\left(\sigma(t)\right)}\left(1 - \frac{p\left(\sigma(t)\right)}{c_1{}^{1-\alpha}}\right)^{\beta} - \frac{e(t)}{z^{\beta}\left(\sigma(t)\right)}\right) - \beta\sigma'(t)\,\omega(t)\frac{z'\left(\sigma(t)\right)}{z\left(\sigma(t)\right)}. \tag{11}$$

Since $z(t) > 0, z'(t) > 0, z''(t) > 0, z'''(t) > 0$, and, $(a(t)(z'''(t))^{\gamma})' < 0$ according to **Lemma 3.1**, we can deduce that $z'(t) \ge \frac{\epsilon}{2}t^2 z'''(t)$

and

$$z'\left(\sigma(t)\right) \ge \frac{\epsilon}{2}\sigma^2(t) z'''\left(\sigma(t)\right) \tag{12}$$

for all $\in \epsilon(0,1)$ and every sufficiently large t. Substituting (12) into (11), we obtain

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)\left(q(t)\frac{1}{m^{\beta}\left(\sigma(t)\right)}\left(1 - \frac{p\left(\sigma(t)\right)}{c_1{}^{1-\alpha}}\right)^{\beta} - \frac{e(t)}{z^{\beta}\left(\sigma(t)\right)}\right) - \frac{\epsilon}{2}\beta\sigma^2(t)\sigma'(t)\frac{z'''\left(\sigma(t)\right)}{z\left(\sigma(t)\right)}\omega(t)$$

$$\omega'(t) \le -\rho(t)\left(q(t)\frac{1}{m^{\beta}\left(\sigma(t)\right)}\left(1 - \frac{p\left(\sigma(t)\right)}{c_1{}^{1-\alpha}}\right)^{\beta} - \frac{e(t)}{z^{\beta}\left(\sigma(t)\right)}\right) + \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\epsilon}{2}\beta\frac{\sigma^2(t)\sigma'(t)}{a^{\frac{1}{\gamma}}\left(\sigma(t)\right)}\frac{a^{\frac{1}{\gamma}}\left(\sigma(t)\right)z'''\left(\sigma(t)\right)}{z\left(\sigma(t)\right)}\omega(t).$$

Since $(a(t)(z''')^{\gamma}(t))' < 0$, we conclude that

$a^{\frac{1}{\gamma}}(t)z'''(t) \le a^{\frac{1}{\gamma}}\left(\sigma(t)\right)z'''\left(\sigma(t)\right).$

Then,

$$\omega'(t) \le -\rho(t)\left(q(t)\frac{1}{m^{\beta}\left(\sigma(t)\right)}\left(1 - \frac{p\left(\sigma(t)\right)}{c_1{}^{1-\alpha}}\right)^{\beta} - \frac{e(t)}{z^{\beta}\left(\sigma(t)\right)}\right) + \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\epsilon\beta}{2}\frac{\sigma^2(t)\sigma'(t)}{a^{\frac{1}{\gamma}}\left(\sigma(t)\right)}\omega(t)\frac{a^{\frac{1}{\gamma}}(t)z'''(t)}{z(\sigma(t)}.$$

$$\omega'(t) \le -\rho(t)\left(q(t)\frac{1}{m^{\beta}\left(\sigma(t)\right)}\left(1 - \frac{p\left(\sigma(t)\right)}{c_1{}^{1-\alpha}}\right)^{\beta} - \frac{e(t)}{z^{\beta}\left(\sigma(t)\right)}\right) + \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\epsilon\beta}{2}\frac{\sigma^2(t)\sigma'(t)}{\left(\rho(t)\,a(\sigma(t)\right)^{\frac{1}{\gamma}}}\left(z(\sigma(t)\right)^{\frac{\beta-\gamma}{\gamma}}\omega^{\frac{\gamma+1}{\gamma}}(t). \tag{13}$$

Because $z'(t) > 0$ and $\beta \ge \gamma$, there exists constants $c_2 > 0$ and $t_2 \ge t_1$ such that $z(\sigma(t)) \ge c_2$

$$z^{\frac{\beta-\gamma}{\gamma}}(\sigma(t)) \ge c_2^{\frac{\beta-\gamma}{\gamma}}, \quad t \ge t_2. \tag{14}$$

Substituting the inequality (14) in (13) gives

$$\omega'(t) \le -\rho(t)\left(q(t)\frac{1}{m^{\beta}\left(\sigma(t)\right)}\left(1 - \frac{p\left(\sigma(t)\right)}{c_1{}^{1-\alpha}}\right)^{\beta} - \frac{e(t)}{c_2^{\beta}}\right) + \frac{\rho'(t)}{\rho(t)}\omega(t) \quad - \frac{\epsilon\gamma c_2^{\frac{\beta-\gamma}{\gamma}}}{2}\frac{\sigma^2(t)\sigma'(t)}{\left(\rho(t)\,a(\sigma(t)\right)^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(t). \tag{15}$$

Using the following inequality in (15),

$$Bu - Au^{\frac{\gamma+1}{\gamma}} \le \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}}, \tag{16}$$

where $A > 0, \quad B \ge 0, \ \gamma > 0$ with

$$B=\frac{\sigma'(t)}{\sigma(t)}, \qquad A=\frac{\epsilon\gamma c_2^{\frac{\beta-\gamma}{\gamma}}\sigma^2(t)\sigma'(t)}{2\left(\rho(t)a(\sigma(t)\right)^{\frac{1}{\gamma}}} \text{ and } \quad u(t)=\omega(t)$$

We get

$$\omega'(t)\le-\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta-\frac{e(t)}{c_2^\beta}\right)+\frac{2^\gamma a(\sigma(t))(\rho'(t))^{\gamma+1}}{(\gamma+1)^{\gamma+1}c_2^{\frac{\beta-\gamma}{\gamma}}(\epsilon\sigma^2(t)\sigma'(t)\rho(t))^\gamma} \tag{17}$$

Integrating (17) from $t_3\ge t_2$ to t, we obtain

$$\int_{t_3}^{t}\left(\begin{array}{c}\rho(s)\left(q(s)\frac{1}{m^\beta(\sigma(s))}\left(1-\frac{p(\sigma(s))}{c_1{}^{1-\alpha}}\right)^\beta-\frac{e(s)}{c_2^\beta}\right)\\ -\frac{2^\gamma a(\sigma(s))(\rho'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}c_2^{\frac{\beta-\gamma}{\gamma}}(\epsilon\sigma^2(s)\sigma'(s)\rho(s))^\gamma}\end{array}\right)ds\le\omega(t_3)$$

This is a contradiction to (3) as $t\to\infty$. Thus the proof is completed.

Theorem 3.2: Suppose that (2) holds. If $0<\beta<\gamma$ and there is a nondecreasing function $\rho\in C'([t_0,\infty),(0,\infty))$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta-\frac{e(t)}{c_3^\beta}\right)-\frac{2^\beta a(t)(\rho'(t))^{\beta+1}}{(\beta+1)^{\beta+1}c_3{}^{\beta-\gamma}(\epsilon\sigma^2(t)\sigma'(t)\rho(t))^\beta}\right)ds=\infty. \tag{18}$$

holds for every $c_1, c_3>0$, then (1) is oscillatory.

Proof: We suppose for contradiction that (1) has an eventually positive solution $x(t)$. As in the proof of Theorem 3.1, the function $\omega(t)$ is defined as in (9) and then (10) holds. By (1), and (7) - (10), we conclude that

$$\omega'(t)\le\frac{\rho'(t)}{\rho(t)}\omega(t)-\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta-\frac{e(t)}{z^\beta(\sigma(t))}\right)\qquad-\beta\sigma'(t)\omega(t)\frac{z'(\sigma(t))}{z(\sigma(t))}.$$

By (12) we observe that

$$\omega'(t)\le-\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta-\frac{e(t)}{z^\beta(\sigma(t))}\right)+\frac{\rho'(t)}{\rho(t)}\omega(t)$$
$$-\frac{\epsilon\beta}{2}\sigma^2(t)\sigma'(t)\omega(t)\left(z'''(t)\right)^{\frac{\beta-\gamma}{\gamma}}\frac{\left(z'''(t)\right)^{\frac{\gamma}{\beta}}}{z(\sigma(t))}\omega(t).$$

$$\omega'(t)\le-\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta-\frac{e(t)}{z^\beta(\sigma(t))}\right)+\frac{\rho'(t)}{\rho(t)}\omega(t)$$
$$-\frac{\epsilon\beta}{2}\frac{\sigma^2(t)\sigma'(t)}{(\rho(t)a(t))^{\frac{1}{\beta}}}\left(z'''(t)\right)^{\frac{\beta-\gamma}{\gamma}}\omega^{\frac{\beta+1}{\beta}}(t). \tag{19}$$

Given that $0<\beta<\gamma$ and (C_1) hold and since $a'(t)\ge0$, it follows that $z''''(t)\le0$. This implies that $z'''(t)$ is nonincreasing. Then there exist constants $c_3>0$ and $t_3\ge t_2$ such that $z'''(t)\le c_3$.

$$\left(z'''(t)\right)^{\frac{\beta-\gamma}{\gamma}}\ge c_3^{\frac{\beta-\gamma}{\gamma}},\quad t\ge t_3. \tag{20}$$

From (19) and (20) it follows that

$$\omega'(t)\le-\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta-\frac{e(t)}{c_3^\beta}\right)+\frac{\rho'(t)}{\rho(t)}\omega(t)\qquad-\frac{\epsilon\beta c_3^{\frac{\beta-\gamma}{\gamma}}}{2}\frac{\sigma^2(t)\sigma'(t)}{(\rho(t)a(t))^{\frac{1}{\beta}}}\omega^{\frac{\beta+1}{\beta}}(t). \tag{21}$$

Using the inequality (16) with

$$B=\frac{\sigma'(t)}{\sigma(t)}, \qquad A=\frac{\epsilon\beta c_3^{\frac{\beta-\gamma}{\gamma}}\sigma^2(t)\sigma'(t)}{2\left(\rho(t)a(\sigma(t)\right)^{\frac{1}{\gamma}}} \text{ and } \quad u(t)=\omega(t)$$

It can be deduced that from (21) that

$$\omega'(t)\le-\rho(t)\left(q(t)\frac{1}{m^\beta(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^\beta-\frac{e(t)}{c_3^\beta}\right)+\frac{2^\beta a(t)(\rho'(t))^{\beta+1}}{(\beta+1)^{\beta+1}c_3{}^{\beta-\gamma}(\epsilon\sigma^2(t)\sigma'(t)\rho(t))^\beta} \tag{22}$$

On integrating (22) over the interval t_4 to t, one arrives at

$$\int_{t_4}^{t}\left(\begin{array}{c}\rho(t)\left(q(t)\dfrac{1}{m^{\beta}(\sigma(t))}\left(1-\dfrac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^{\beta}-\dfrac{e(t)}{c_3^{\beta}}\right)\\ -\dfrac{2^{\beta}a(t)(\rho'(t))^{\beta+1}}{(\beta+1)^{\beta+1}c_3{}^{\beta-\gamma}(\epsilon\sigma^2(t)\sigma'(t)\rho(t))^{\beta}}\end{array}\right)ds\le\omega(t_4)$$

This is a contradiction to (18) as $t\to\infty$. Thus the proof is completed.

Theorem 3.3: Assume that (2) holds. If $\beta\ge\gamma$ and there is a nondecreasing function $\rho\in C'([t_0,\infty),(0,\infty))$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\begin{array}{c}M_1^{\beta-\gamma}\rho_1(s)\left(q(s)\dfrac{1}{m^{\beta}(\sigma(s))}\left(1-\dfrac{p(\sigma(s))}{c_1{}^{1-\alpha}}\right)^{\beta}-\dfrac{e(s)}{M_1^{\beta}}\right)\left(\dfrac{\sigma^3(s)}{s}\right)^{\frac{3\beta}{\epsilon}}\\ -\dfrac{2^{\gamma}}{(\gamma+1)^{\gamma+1}}\dfrac{a(s)(\sigma_1'(s))^{\gamma+1}}{(\epsilon s^2\sigma_1(s))^{\gamma}}\end{array}\right)ds=\infty. \tag{23}$$

holds for every $c_1,c_2>0$, then (1) is oscillatory.

Proof. We suppose for contradiction that (1) has an eventually positive solution. Now we define a function

$$\omega_1(t)=\rho_1(t)\frac{a(t)(z'''(t))^{\gamma}}{z^{\gamma}(t)} \tag{24}$$

which yields $\omega_1(t)>0$, and

$$\omega_1'(t)=\rho_1'(t)\frac{a(t)(z'''(t))^{\gamma}}{z^{\gamma}(t)}+\rho_1(t)\frac{(a(t)(z'''(t))^{\gamma})'}{z^{\gamma}(t)}-\gamma\rho_1(t)\frac{a(t)(z'''(t))^{\gamma}z'(t)}{z^{\beta+1}(t)} \tag{25}$$

From (1), (7), (8), (24) and (25) that

$$\omega_1'(t)\le-\rho_1(t)\left(q(t)\frac{1}{m^{\beta}(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^{\beta}-\frac{e(t)}{z^{\beta}(\sigma(t))}\right)\frac{z^{\beta}(\sigma(t))}{z^{\gamma}(t)}+\frac{\rho_1'(t)}{\rho_1(t)}\omega_1(t)-\gamma\frac{z'(t)}{z(t)}\omega_1(t). \tag{26}$$

We deduce from **Lemma 3.2**, that

$$z(t)\ge\frac{\epsilon}{3}tz'(t),$$

and hence,

$$\frac{z(\sigma(t))}{z(t)}\ge\left(\frac{\sigma^3(t)}{t}\right)^{\frac{3}{\epsilon}} \tag{27}$$

From **Lemma 3.1**, we conclude that

$$z'(t)\ge\frac{\epsilon}{2}t^2z'''(t) \tag{28}$$

for all $\epsilon\in(0,1)$. Thus, by (26), (27) and (28), we have

$$\omega_1'(t)\le-\rho_1(t)\left(q(t)\frac{1}{m^{\beta}(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^{\beta}-\frac{e(t)}{z^{\beta}(\sigma(t))}\right)z^{\beta-\gamma}(t)\frac{z^{\beta}(\sigma(t))}{z^{\beta}(t)}+\frac{\rho_1'(t)}{\rho_1(t)}\omega_1(t)-\frac{\epsilon\gamma}{2}t^2\frac{z'''(t)}{z(t)}\omega_1(t).$$

$$\omega_1'(t)=-\rho_1(t)\left(q(t)\frac{1}{m^{\beta}(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1{}^{1-\alpha}}\right)^{\beta}-\frac{e(t)}{z^{\beta}(\sigma(t))}\right)z^{\beta-\gamma}(t)\left(\frac{\sigma^3(t)}{t}\right)^{\frac{3\beta}{\epsilon}}+\frac{\rho_1'(t)}{\rho_1(t)}\omega_1(t)-\frac{\epsilon\gamma}{2}\frac{t^2}{(a(t)\sigma_1(t))^{\frac{1}{\gamma}}}\omega^{\frac{1+\gamma}{\gamma}}{}_1(t). \tag{29}$$

Since $z'(t)>0$, there exists a $t_3\ge t_2$ and a constant $M_1>0$ such that

$$z(t)>M_1.$$

Since $\beta\ge\gamma$, then

$$z^{\beta-\gamma}(t)>M_1^{\beta-\gamma} \tag{30}.$$

Thus inequality (29) gives

$$\omega_1'(t) = -M_1^{\beta-\gamma}\rho_1(t)\left(q(t)\frac{1}{m^{\beta}(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1^{1-\alpha}}\right)^{\beta}-\frac{e(t)}{M_1^{\beta}}\right)\left(\frac{\sigma^3(t)}{t}\right)^{\frac{3\beta}{\epsilon}}+\frac{\rho_1'(t)}{\rho_1(t)}\omega_1(t) - \frac{\epsilon\gamma t^2}{2(a(t)\sigma_1(t))^{\frac{1}{\gamma}}}\omega^{\frac{1+\gamma}{\gamma}}{}_1(t). \tag{31}$$

Using the inequality (16) with

$$B=\frac{\rho_1'(t)}{\rho_1(t)},\quad A=\frac{\epsilon\gamma t^2}{2(a(t)\sigma_1(t))^{\frac{1}{\gamma}}} \text{ and } u(t)=\omega_1(t)$$

we can derive the following inequality

$$\omega_1'(t) = -M_1^{\beta-\gamma}\rho_1(t)\left(q(t)\frac{1}{m^{\beta}(\sigma(t))}\left(1-\frac{p(\sigma(t))}{c_1^{1-\alpha}}\right)^{\beta}-\frac{e(t)}{M_1^{\beta}}\right)\left(\frac{\sigma^3(t)}{t}\right)^{\frac{3\beta}{\epsilon}}+\frac{2^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{a(t)(\sigma_1'(t))^{\gamma+1}}{(\epsilon t^2\sigma_1(t))^{\gamma}} \tag{32}$$

On integrating (32) from $t_2 \geq t$ to t, we get

$$\int_{t_2}^{t}\left(M_1^{\beta-\gamma}\rho_1(s)\left(q(s)\frac{1}{m^{\beta}(\sigma(s))}\left(1-\frac{p(\sigma(s))}{c_1^{1-\alpha}}\right)^{\beta}-\frac{e(s)}{M_1^{\beta}}\right)\left(\frac{\sigma^3(s)}{s}\right)^{\frac{3\beta}{\epsilon}}-\frac{2^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{a(s)(\sigma_1'(s))^{\gamma+1}}{(\epsilon s^2\sigma_1(s))^{\gamma}}\right)ds \leq \omega_1(t_2),$$

which contradicts (23) as $t \to \infty$. Thus the proof is completed.

4. Conclusion

The goal of this paper is to study the "Oscillatory behavior of fourth-order Emden-Fowler differential equations with a sub-linear neutral term" of equation (1) by using Riccati Transformation technique. Further extension of these results can be used to study a class of system of higher order Neutral differential equations as well as Fractional order equations. Some research in this area is in progress.

The results of this study complement many of previously published findings in the literature. To our knowledge, this equation has not been studied by many researchers, so it would be a good idea to apply these results to non-linear higher-order NDEs in the future.

5. References

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